

# A SIMPLICIAL COMPLEX OF NICHOLS ALGEBRAS

M. CUNTZ AND S. LENTNER

**ABSTRACT.** We translate the concept of restriction of an arrangement in terms of Hopf algebras. In consequence, every Nichols algebra gives rise to a simplicial complex decorated by Nichols algebras with restricted root systems. As applications, some of these Nichols algebras provide Weyl groupoids which do not arise for diagonal Nichols algebras and in fact we realize all root systems of finite Weyl groupoids of rank greater than three. Further, our result explains the root systems of the folded Nichols algebras over nonabelian groups and of generalized Satake diagrams.

## CONTENTS

1. Introduction	2
2. Weyl groupoids and crystallographic arrangements	7
2.1. Simplicial arrangements	7
2.2. Crystallographic arrangements	8
2.3. Cartan graphs and Weyl groupoids	9
2.4. Classification of finite Weyl groupoids	11
3. Restrictions of arrangements and root systems	12
3.1. Parabolic restriction	13
3.2. All parabolic restrictions of finite root systems	14
3.3. Folding restriction	15
4. Arrangements and Nichols algebras	21
4.1. Nichols algebras	21
4.2. The Weyl groupoid of a Nichols algebra	23
4.3. Restrictions of Nichols algebras	32
4.4. First examples of restrictions	35
4.5. Nichols algebras with new Weyl groupoids	38
5. A simplicial complex of Nichols algebras	41

---

S. Lentner is partly supported by the DFG Research Training Group 1670. Most results of this article were achieved at meetings supported by the DFG within the priority programme 1388.

5.1. The simplicial complex	41
5.2. The decoration by Nichols algebras in braided categories	42
5.3. Example: A Nichols algebra of rank 3	44
References	47

## 1. INTRODUCTION

Nichols algebras are Hopf algebras in a braided category, that enjoy certain universal properties and appear naturally in every pointed Hopf algebra. The most prominent examples are the Borel parts  $u_q(\mathfrak{g})^\pm$  of the small quantum groups. In fact the Lie-theoretical flavour is retained in general, and we know from [AHS10] that any Nichols algebra is controlled by a Weyl groupoid. This is a natural generalization of a Weyl group, where different Dynkin diagrams are attached to the different groupoid objects. Since [Cun11] we have a good perception of a Weyl groupoid as an arrangement of hyperplanes, and in [CH15] all finite Weyl groupoids were classified: There are infinitely many finite Weyl groupoids of rank 2, an additional infinite series between  $D_n$  and  $C_n$ , as well as 74 sporadic cases up to rank 8.

However, compared to the theory of semisimple Lie algebras where root systems provide a complete classification (at least over the complex numbers), for Nichols algebras the situation is more complicated: There are non-isomorphic Nichols algebras whose corresponding Weyl groupoids are equivalent, and it is not known yet whether all Weyl groupoids arise as symmetry structures of Nichols algebras (not even in the case of finite Weyl groupoids). The latter problem has been brought up on several occasions, notably by the first author at the Oberwolfach Mini-Workshop on “Nichols Algebras and Weyl Groupoids” in 2012:

*Question 1.1.* Is there a Weyl groupoid which does not occur as symmetry structure of a Nichols algebra?<sup>1</sup> Is there a characterization of those root systems that cannot appear as root systems of Nichols algebras?

One way to answer part of these questions is to introduce constructions producing new Nichols algebras (and hopefully new Weyl groupoids as well). For example, *folding of root systems* has been investigated by the second author to construct new large rank Nichols algebras over nonabelian groups [Len14a]. As proven recently by Heckenberger and Vendramin [HV14], these Nichols algebras are the only

---

<sup>1</sup>For Nichols algebras over groups, not all Weyl groupoids appear.

examples in large rank and characteristic zero. There are four exceptions of rank two and three which are not obtained by this folding construction, and for rank one the classification is still largely open. The second author hence asked at the Oberwolfach Mini-Workshop on “Infinite dimensional Hopf algebras” in 2014:

*Question 1.2.* Systematically explain the impact of the folding construction on the root system<sup>2</sup>. Are there more general folding constructions on the root system of a Nichols algebra? Is there a closed construction of all Nichols algebras over nonabelian groups in this way?

In this article we discuss such a construction, which partly answers the previous questions, namely *restriction* on the arrangement of hyperplanes given by the root system. More precisely we proceed as follows:

The set  $\mathcal{A}$  of hyperplanes  $\alpha^\perp \subset V$  orthogonal to the roots is a *crystallographic arrangement* and this induces an equivalence between crystallographic arrangements and finite root systems. For details see the preliminaries in Section 2.

Let  $X$  be a subspace of  $V$ . Then the restriction  $\mathcal{A}^X$  is the set of hyperplanes in  $\mathcal{A}$  of the form  $X \cap H$  for  $H \in \mathcal{A}$ . This construction is discussed in Section 3. Typically, not all hyperplanes  $H \in \mathcal{A}$  give rise to hyperplanes  $X \cap H$  in  $\mathcal{A}^X$ , and several hyperplanes in  $\mathcal{A}$  may give rise to the same hyperplane in  $\mathcal{A}^X$ .

There are two important special cases:

- The case when  $X$  is an intersection of some hyperplanes of  $\mathcal{A}$ , we call this *parabolic restriction*. Then  $\mathcal{A}^X$  is automatically again crystallographic. In fact, most finite Weyl groupoids appear in this way, including examples which are not attained from Nichols algebras over groups.
- The case when  $X$  is the fixpoint set of an automorphism  $g$  of  $\mathcal{A}$ . In general,  $\mathcal{A}^X$  must not be crystallographic, but in several cases it is. In the special case where  $g$  permutes a set of simple roots (called *permutation restriction*), these restrictions describe the root system of the folded Nichols algebra over the centrally extended group.

In Section 4 we turn our attention to Weyl groupoids of Nichols algebras. We start with a Nichols algebra  $\mathcal{B}(M)$ , which is typically diagonal. Let  $J \subset I$  be a

---

<sup>2</sup>This has previously been calculated by hand.

subset of the simple roots for  $\mathcal{B}(M)$ , denote by  $M_J \subset M$  the corresponding sub-Yetter-Drinfel'd module and consider the associated Nichols algebra of coinvariants

$$\mathcal{B}(\bar{M}) := \mathcal{B}(M)^{\text{coin}(\mathcal{B}(M_J))}.$$

This is a Nichols algebra in the category of  $\mathcal{B}(M_J)$ -Yetter-Drinfel'd modules. The key result of this article in Theorem 4.14 is that the root system of  $\mathcal{B}(\bar{M})$  is the parabolic restriction  $\mathcal{A}^X$ ,  $X = J^\perp$ . Moreover,  $\mathcal{B}(M_J)$  (determining the category) corresponds precisely to the sub-arrangement of hyperplanes in  $\mathcal{A}$  that do not give rise to hyperplanes in  $\mathcal{A}^X$ , while the number of different hyperplanes in  $\mathcal{A}$  that give rise to the same hyperplane in  $\mathcal{A}^X$ , determines the dimensions of the Yetter-Drinfel'd modules in the PBW-basis of  $\mathcal{B}(\bar{M})$ .

In Section 5 we conclude by viewing the set of all parabolic restrictions of a Nichols algebra as a simplicial complex. In this picture, the reflection operation for Nichols algebras has a particularly nice interpretation in terms of restrictions and dualization.

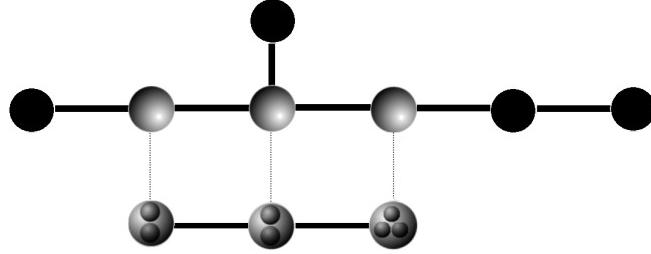
We now discuss *applications*, most of which are new and interesting already for the case of a diagonal Nichols algebra and even for the special case of a quantum group  $u_q(\mathfrak{g})^+$  associated to a Lie algebra  $\mathfrak{g}$ :

As first result, we obtain Nichols algebras with Weyl groupoids, which were not attained from a previously known Nichols algebra over a group:

**Theorem (4.17).**

- (1) *There exist Nichols algebras whose corresponding crystallographic arrangements are the sporadic arrangements of rank three labeled 7, 13, 14, 15, 20, 23, although these Nichols algebras do not exist over any finite group.*
- (2) *Since every crystallographic arrangement of rank greater than three is a restriction of a Weyl arrangement, every crystallographic arrangement of rank greater than three is symmetry structure of some Nichols algebra.*

*Example (4.19).* Let  $\mathcal{B}(M) = u_q(E_7)^+$ , which is a diagonal Nichols algebra of dimension  $\text{ord}(q^2)^{63}$ . We consider the parabolic restriction indicated in the diagram:



Then the restriction of the Weyl group arrangement  $E_7$  with 63 roots has a Weyl groupoid of sporadic type 7 and 13 roots (one of which is nonreduced), namely

$R_+^a$	$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$	$(1, 1, 0)$	$(0, 1, 1)$	$(1, 1, 1)$	$+2(1, 1, 1)$
multiplicity	2	2	3	4	6	12	+3
$R_+^a$	$(1, 2, 1)$	$(2, 2, 1)$	$(1, 2, 2)$	$(2, 3, 2)$	$(2, 3, 3)$	$(2, 4, 3)$	$(3, 4, 3)$
multiplicity	6	3	6	6	2	1	2

This yields a Nichols algebra  $\mathcal{B}(\bar{M})$  of dimension  $\text{ord}(q^2)^{58}$  over an object  $\bar{M}$  of rank 3 and dimension  $2 + 2 + 3$  in the braided category  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M_J)}\mathcal{YD}$  with  $J = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7\}$ , where  $\mathcal{B}(M_J)$  is an ordinary Borel part  $u_q(A_1 \times A_1 \times A_2)^+$ .

This root system is clearly not one of Lie type. This means that reflections in the Weyl groupoid connect the Weyl chamber described above with other chambers corresponding possibly to different Dynkin diagrams and non-isomorphic Nichols algebras (compare Example 4.8).

For the restriction these different choices arise from different parabolics  $J'$  of  $E_7$  in the Weyl group orbit of the choice  $J$  above. In the main text, this is discussed in more detail.

As a second application, we note that quantum groups  $u_q(\mathfrak{g})$  (and other pointed Hopf algebras [AY13]) can be obtained from a Drinfel'd double construction. Our restricted Nichols algebra  $\mathcal{B}(\bar{M})$  exists in the category  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M_J)}\mathcal{YD}$ , hence it is roughly a Nichols algebra in the category of  $u_q(\mathfrak{g}_J)$ -modules, where  $\mathfrak{g}_J$  is the Lie subalgebra of  $\mathfrak{g}$  associated to the subset of simple roots  $J$ . As an example<sup>3</sup> with  $J = I \setminus \{\alpha_1\}$

---

<sup>3</sup>We are thankful to N. Andruskiewitsch for pointing out this source to us.

of type  $A_{n-1}$ , this explains the  $\mathfrak{sl}_n$ -representations in [Rosso98, Sec. 4]. See our Example 4.15.

*Question 1.3.* Can Nichols algebras  $\mathcal{B}(\bar{M})$  in the category of modules over a quantum double of some  $\mathcal{B}(M_J)$  and in a second step over  $u_q(\mathfrak{g}_J)$  be characterized by restrictions of ordinary Nichols algebras  $\mathcal{B}(M)$ ,  $M \supset M_J$ ? This would be the first Nichols algebras over non-semisimple Hopf algebras, and it would also be interesting from the perspective of logarithmic conformal field theory.

A third application is in the context of quantum symmetric pairs: Our graphical representation using black dots is suggested by *Satake diagrams* of Lie algebras<sup>4</sup>. Indeed, we define *folding* of a crystallographic arrangement (resp. the root system of a Weyl groupoid) as the restriction to the fixed-point set  $X$  of some automorphism group of the root system, and we prove a respective Lemma 3.9 that enables us to classify such scenarios again by Satake diagrams.

*Example 1.4.* The previous example shows that we even get new Satake diagrams for ordinary Lie algebras, here  $E_7$  over a parabolic subalgebra  $A_1 \times A_1 \times A_2$ . This would be excluded by the classical theory, because the restricted root system (relative root system, quotient root system) corresponds to a Weyl groupoid.

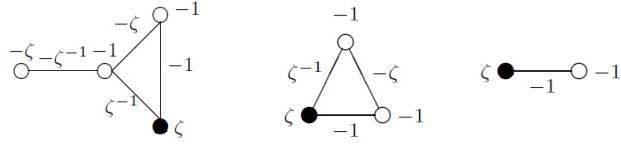
Satake diagrams for Nichols algebras of Lie type appear prominently in the theory of quantum symmetric pairs, introduced by Noumi, Sugitani, and Dijkhuizen for  $\mathfrak{g}$  of classical type in connection with the reflection equation and in theoretical physics (see e.g. [NS95]) and independently by G. Letzter from the Iwasawa decomposition for the quantum group (see e.g. [Let97]), generalized to Kac Moody algebras by [Kolb14]. Our results show that one can consider more general Satake diagrams for Lie type root systems, which do not yield Lie type root systems anymore, as in the example above. On the other hand they allow Satake diagrams to be considered for arbitrary Nichols algebras, in particular for super Lie algebras and color Lie algebras.

As a final outlook, this may point to a more general folding construction for Nichols algebras than in [Len14a], since in the known cases the restricted root system  $\mathcal{A}^X$  reproduces the correct folded root system. As a new Example 4.13 we

---

<sup>4</sup>We thank S. Kolb for having brought up this topic to our attention at the Oberwolfach miniworkshop mentioned above.

identify an inclusion chain of three Nichols algebra, for which restriction would reproduce the correct root system and root space dimensions for the exceptional Nichols algebras of rank 3, 2, 1 over nonabelian groups involving  $\mathbb{S}_3$  in [HV14]:



In this article we do *not* attempt to define non-parabolic restriction of Nichols algebras, that should come with an additional group action on the root spaces  $M_\alpha$ . Note however that in the theory of quantum symmetric pairs there are explicit methods to construct coideal subalgebras corresponding to the fixed-point set of the root system automorphism. This might give the right inspiration for constructing the respective restricted Nichols algebra in the general (non-parabolic, non-permutation) case.

## 2. WEYL GROUPOIDS AND CRYSTALLOGRAPHIC ARRANGEMENTS

**2.1. Simplicial arrangements.** Let  $r \in \mathbb{N}$ ,  $V := \mathbb{R}^r$ . For  $\alpha \in V^*$  we write  $\alpha^\perp = \ker(\alpha)$ . We first recall the definition of a simplicial arrangement (compare [OT92, 1.2, 5.1]).

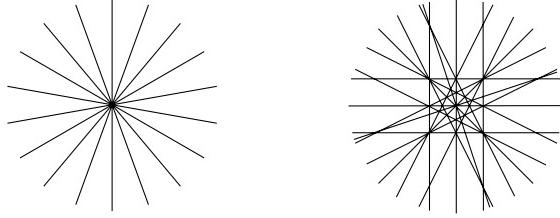
*Definition 2.1.* An *arrangement of hyperplanes*  $\mathcal{A}$  is a finite set of hyperplanes in  $V$ . Let  $\mathcal{K}(\mathcal{A})$  be the set of connected components (*chambers*) of  $V \setminus \bigcup_{H \in \mathcal{A}} H$ . If every chamber  $K$  is an *open simplicial cone*, i.e. there exist  $\alpha_1^\vee, \dots, \alpha_r^\vee \in V$  such that

$$K = \left\{ \sum_{i=1}^r a_i \alpha_i^\vee \mid a_i > 0 \text{ for all } i = 1, \dots, r \right\} =: \langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle_{>0},$$

then  $\mathcal{A}$  is called a *simplicial arrangement*.

*Example 2.2.*

- (1) The picture on the left is a simplicial arrangement in  $\mathbb{R}^2$ . The picture on the right is a representation of a three dimensional simplicial arrangement in the projective plane. The simplicial cones become triangles in this representation.



- (2) Let  $W$  be a real reflection group,  $R$  the set of roots of  $W$ . Then  $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$  is a simplicial arrangement.

**2.2. Crystallographic arrangements.** Let  $\mathcal{A} = \{H_1, \dots, H_n\}$ ,  $|\mathcal{A}| = n$  be a simplicial arrangement. For each  $H_i$ ,  $i = 1, \dots, n$  we choose an element  $x_i \in V^*$  such that  $H_i = x_i^\perp$  and let  $R := \{\pm x_1, \dots, \pm x_n\} \subseteq V^*$ .

Since  $\mathcal{A}$  is simplicial, for each chamber  $K \in \mathcal{K}(\mathcal{A})$  there is a unique choice of a basis  $\{\alpha_1^\vee, \dots, \alpha_r^\vee\}$  such that  $K = \langle \alpha_1^\vee, \dots, \alpha_r^\vee \rangle_{>0}$  and such that its dual basis  $\{\alpha_1, \dots, \alpha_r\}$  is a subset of  $R$ . We write

$$\begin{aligned}\Delta^K &= \{\alpha_1, \dots, \alpha_r\} \\ &= \{ \text{“normal vectors in } R \text{ of the walls of } K \text{ pointing to the inside”} \}.\end{aligned}$$

We are now ready for the main definition.

*Definition 2.3.* Let  $\mathcal{A}$  be a simplicial arrangement and  $R \subseteq V^*$  a finite set such that  $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$  and  $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$  for all  $\alpha \in R$ . We call  $(\mathcal{A}, R)$  a *crystallographic arrangement* if for all  $K \in \mathcal{K}(\mathcal{A})$ :

$$R \subseteq \sum_{\alpha \in \Delta^K} \mathbb{Z}\alpha.$$

Two crystallographic arrangements  $(\mathcal{A}, R)$ ,  $(\mathcal{A}', \bar{R})$  in  $V$  are called *equivalent* if there exists  $\psi \in \text{Aut}(V^*)$  with  $\psi(R) = \bar{R}$ . We then write  $(\mathcal{A}, R) \cong (\mathcal{A}', \bar{R})$ .

*Example 2.4.* (1) Let  $R$  be the set of roots of the root system of a crystallographic Coxeter group. Then  $(\{\alpha^\perp \mid \alpha \in R\}, R)$  is a crystallographic arrangement.

- (2) If  $R_+ := \{(1, 0), (3, 1), (2, 1), (5, 3), (3, 2), (1, 1), (0, 1)\}$ , then  $(\{\alpha^\perp \mid \alpha \in R_+\}, R_+ \dot{\cup} -R_+)$  is a crystallographic arrangement.

As most crystallographic arrangements are not defined by reflection groups, they have less symmetries in general. As a substitute for this large symmetry group, it turns out that a corresponding *Weyl groupoid* is the right symmetry structure.

**2.3. Cartan graphs and Weyl groupoids.** We now define the notion of a Weyl groupoid which was introduced by Heckenberger and Yamane [HY08] and reformulated in [CH09]. But before we present the axioms, let us look at an example.

*Example 2.5.* Let  $\alpha_1 = (1, 0, 0)$ ,  $\alpha_2 = (0, 1, 0)$ ,  $\alpha_3 = (0, 0, 1)$ , and

$$R_+^a := \{\alpha_1, \alpha_2, \alpha_3, (0, 1, 1), (0, 1, 2), (1, 0, 1), (1, 1, 1), (1, 1, 2)\}.$$

For  $1 \leq i, j \leq 3$ , define entries  $c_{i,j}$  of a matrix  $C$  by

$$c_{i,j} := -\max\{k \mid k\alpha_i + \alpha_j \in R_+^a\}, \quad c_{i,i} := 2 \quad \text{for } i \neq j,$$

thus

$$C^a = (c_{i,j})_{i,j} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

This is a *generalized Cartan matrix*. It defines *reflections* via

$$\sigma_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i \quad \text{for } j = 1, 2, 3.$$

For instance,

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\sigma_1(R_+^a) = \{-\alpha_1, \alpha_2, (1, 0, 1), (1, 1, 1), (2, 1, 2), \alpha_3, (0, 1, 1), (1, 1, 2)\}.$$

The elements of  $\sigma_1(R_+^a)$  are *positive* or *negative*. Let  $R^a = R_+^a \cup -R_+^a$  and  $R^b = \sigma_1(R^a) =: R_+^b \cup -R_+^b$ . Again, one can construct a Cartan matrix from  $R_+^b$  and it gives new reflections. In this example, we obtain a diagram:

$$\begin{array}{ccc} \left( \begin{array}{rrr} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{array} \right) & \xrightarrow{\sigma_3} & \left( \begin{array}{rrr} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{array} \right) & \xrightarrow{\sigma_2} & \left( \begin{array}{rrr} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{array} \right) \\ \sigma_1 \Big| & & & & \Big| \sigma_1 \\ \left( \begin{array}{rrr} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{array} \right) & & & & \left( \begin{array}{rrr} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{array} \right) \end{array}$$

For the general definition, we first recall:

*Definition 2.6.* Let  $I := \{1, \dots, r\}$  and  $\{\alpha_i \mid i \in I\}$  the standard basis of  $\mathbb{Z}^I$ . A *generalized Cartan matrix*  $C = (c_{ij})_{i,j \in I}$  is a matrix in  $\mathbb{Z}^{I \times I}$  such that

- (M1)  $c_{ii} = 2$  and  $c_{jk} \leq 0$  for all  $i, j, k \in I$  with  $j \neq k$ ,
- (M2) if  $i, j \in I$  and  $c_{ij} = 0$ , then  $c_{ji} = 0$ .

The above diagram of matrices is a *Cartan graph*<sup>5</sup>:

*Definition 2.7.* Let  $A$  be a non-empty set,  $\rho_i : A \rightarrow A$  a map for all  $i \in I$ , and  $C^a = (c_{jk}^a)_{j,k \in I}$  a generalized Cartan matrix in  $\mathbb{Z}^{I \times I}$  for all  $a \in A$ . The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$$

is called a *Cartan graph* if

- (C1)  $\rho_i^2 = \text{id}$  for all  $i \in I$ ,
- (C2)  $c_{ij}^a = c_{ij}^{\rho_i(a)}$  for all  $a \in A$  and  $i, j \in I$ .

*Definition 2.8.* Let  $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$  be a Cartan graph. For all  $i \in I$  and  $a \in A$  define  $\sigma_i^a \in \text{Aut}(\mathbb{Z}^I)$  by

$$(1) \quad \sigma_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \quad \text{for all } j \in I.$$

The *Weyl groupoid* of  $\mathcal{C}$  is the category  $\mathcal{W}(\mathcal{C})$  such that  $\text{Ob}(\mathcal{W}(\mathcal{C})) = A$  and the morphisms are compositions of maps  $\sigma_i^a$  with  $i \in I$  and  $a \in A$ , where  $\sigma_i^a$  is considered as an element in  $\text{Hom}(a, \rho_i(a))$ . The cardinality of  $I$  is the *rank of*  $\mathcal{W}(\mathcal{C})$ .

*Definition 2.9.* A Cartan graph is called *connected* if its Weyl groupoid is connected, that is, if for all  $a, b \in A$  there exists  $w \in \text{Hom}(a, b)$ . The Cartan graph is called *simply connected*, if  $\text{Hom}(a, a) = \{\text{id}^a\}$  for all  $a \in A$ . There is a straight forward notion of *equivalence* of Cartan graphs which we skip here.

Let  $\mathcal{C}$  be a Cartan graph. For all  $a \in A$  let

$$(R^{\text{re}})^a = \{\text{id}^a \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \mid k \in \mathbb{N}_0, i_1, \dots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set  $(R^{\text{re}})^a$  are called *real roots* (at  $a$ ). The pair  $(\mathcal{C}, ((R^{\text{re}})^a)_{a \in A})$  is denoted by  $\mathcal{R}^{\text{re}}(\mathcal{C})$ . A real root  $\alpha \in (R^{\text{re}})^a$ , where  $a \in A$ , is called positive (resp. negative) if  $\alpha \in \mathbb{N}_0^I$  (resp.  $\alpha \in -\mathbb{N}_0^I$ ).

*Definition 2.10.* Let  $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$  be a Cartan scheme. For all  $a \in A$  let  $R^a \subseteq \mathbb{Z}^I$ , and define  $m_{i,j}^a = |R^a \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|$  for all  $i, j \in I$  and  $a \in A$ . We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$$

---

<sup>5</sup>In earlier papers, Cartan graphs were called Cartan schemes. The new term was chosen by Andruskiewitsch, Heckenberger and Schneider to avoid confusion with geometric schemes.

is a *root system of type  $\mathcal{C}$* , if it satisfies the following axioms.

- (R1)  $R^a = R_+^a \cup -R_+^a$ , where  $R_+^a = R^a \cap \mathbb{N}_0^I$ , for all  $a \in A$ .
- (R2)  $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$  for all  $i \in I$ ,  $a \in A$ .
- (R3)  $\sigma_i^a(R^a) = R^{\rho_i(a)}$  for all  $i \in I$ ,  $a \in A$ .
- (R4) If  $i, j \in I$  and  $a \in A$  such that  $i \neq j$  and  $m_{i,j}^a$  is finite, then  $(\rho_i \rho_j)^{m_{i,j}^a}(a) = a$ .

The root system  $\mathcal{R}$  is called *finite* if for all  $a \in A$  the set  $R^a$  is finite. By [CH09, Prop. 2.12], if  $\mathcal{R}$  is a finite root system of type  $\mathcal{C}$ , then  $\mathcal{R} = \mathcal{R}^{\text{re}}$ , and hence  $\mathcal{R}^{\text{re}}$  is a root system of type  $\mathcal{C}$  in that case.

*Remark 2.11.* If  $\mathcal{C}$  is a Cartan graph and there exists a root system of type  $\mathcal{C}$ , then  $\mathcal{C}$  satisfies

- (C3) If  $a, b \in A$  and  $\text{id} \in \text{Hom}(a, b)$ , then  $a = b$ .

*Example 2.12 (Lie type).* Let  $\mathfrak{g}$  be a semisimple finite-dimensional complex Lie algebra. Then this is uniquely determined (up to isomorphisms) by its corresponding root system, which is the root system of a finite Weyl group  $W$ . The corresponding Cartan graph has exactly one object  $a$  where  $C^a$  is the Cartan matrix of  $W$ . The set  $R^a$  is the root system of  $W$ .

**2.4. Classification of finite Weyl groupoids.** Connected simply connected Cartan graphs for which the real roots are a finite root system (these are also called *universal finite Weyl groupoids*) are in one-to-one correspondence with crystallographic arrangements. Under the correspondence, every chamber  $K$  of an arrangement corresponds to an object  $a$ ; the set  $R^a$  consists of the coordinates of  $R$  with respect to the basis  $\Delta^K$ .

**Theorem 2.13** (see [Cun11]). *Let  $\mathfrak{A}$  be the set of all crystallographic arrangements and  $\mathfrak{C}$  be the set of all connected simply connected Cartan graphs for which the real roots are a finite root system. Then the map*

$$\mathfrak{C}/\cong \rightarrow \mathfrak{A}/\cong, \quad \overline{\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})} \mapsto \overline{(\{\alpha^\perp \mid \alpha \in R^a\}, R^a)},$$

where  $a$  is any object of  $\mathcal{C}$ , is a bijection.

A series of five papers by the first author and Heckenberger culminates in the following complete classification of finite Weyl groupoids and thus equivalently of crystallographic arrangements.

**Theorem 2.14** (see [CH15]). *There are exactly three families of irreducible crystallographic arrangements:*

- (1) *The family of rank two parametrized by triangulations of convex  $n$ -gons by non-intersecting diagonals.*
- (2) *For each rank  $r > 2$ , arrangements of type  $A_r$ ,  $B_r$ ,  $C_r$  and  $D_r$ , and a further series of  $r - 1$  arrangements denoted  $\mathcal{A}_r^k(2)$  in [OT92, 6.4].*
- (3) *Further 74 “sporadic” arrangements of rank  $r$ ,  $3 \leq r \leq 8$ .*

*Remark 2.15.* Theorem 2.14 classifies simply-connected Cartan graphs. Every Cartan graph has a simply-connected cover. For example the Cartan graph of Lie type has a single Cartan matrix and the Weyl groupoid has a single object with automorphism group the Weyl group  $W$ . The simply-connected cover has  $|W|$  Cartan matrices, all of the same type, and its Weyl groupoid has  $|W|$  objects and no non-trivial automorphisms.

Automorphism groups and minimal quotients of finite Weyl groupoids have all been determined in [CH15]. Whether one can consider such non-simply-connected quotients depends on additional data in the application. For example, the Lie algebra  $\mathfrak{sl}_3$  resp. the Lie superalgebra  $\mathfrak{sl}(2|1)$  have the same associated arrangement  $A_2$ , but in the first case one usually considers the Weyl group (one object), while in the latter case some roots are labeled differently, so there are two types of chambers.

### 3. RESTRICTIONS OF ARRANGEMENTS AND ROOT SYSTEMS

*Definition 3.1* ([OT92, 1.12-1.14]). Let  $\mathcal{A}$  be an arrangement in  $V$ . We denote  $L(\mathcal{A})$  the set of all nonempty intersections of elements of  $\mathcal{A}$ .

For a subspace  $X \leq V$ , define a subarrangement  $\mathcal{A}_X$  of  $\mathcal{A}$  called the *localization at  $X$*  by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Define an arrangement  $\mathcal{A}^X$  in  $X$  called the *restriction to  $X$*  by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

If  $\mathcal{A}$  is a reflection arrangement of a Coxeter group  $W$ , then localizations of  $\mathcal{A}$  are the reflection arrangements of parabolic subgroups of  $W$ . Thus localizations are easy to understand from the algebraic point of view. Restrictions  $\mathcal{A}^X$  however are not reflection arrangements in general, even in the case when  $X$  is in the intersection lattice  $L(\mathcal{A})$ .

However, the restrictions of a crystallographic arrangement to an element  $X \in L(\mathcal{A})$  is always again a crystallographic arrangement. This yields an important class of examples:

**3.1. Parabolic restriction.** In this subsection we discuss the following case of restriction:

*Definition 3.2.* A *parabolic restriction* of an arrangement  $\mathcal{A}$  is a restriction  $\mathcal{A}^X$  to an intersection of existing hyperplanes  $X \in L(\mathcal{A})$ .

In contrast to parabolic localization  $\mathcal{A}_X$  this corresponds to *quotienting out* a parabolic subgroup and the result is in general not a reflection arrangement. Note that any parabolic restriction to an intersection of hyperplanes  $X \in L(\mathcal{A})$  can be obtained by repeatedly restricting to one hyperplane  $H \in \mathcal{A} \subset L(\mathcal{A})$ .

It is an easy fact that (see [CRT12], [BC12]):

**Lemma 3.3.** Let  $(\mathcal{A}, R)$  by a crystallographic arrangement, then any parabolic restriction  $\mathcal{A}^X$  to some  $X \in L(\mathcal{A})$  is again a crystallographic arrangement.

More precisely, a root system  $\bar{R}$  for  $\mathcal{A}^X$  is given as follows: Suppose without loss of generality  $X \in \mathcal{A}$  a single hyperplane and a chamber  $K$  chosen adjacent to  $X$ , say by suitable numbering  $X = \alpha_1^\perp$ . Then the restriction of  $\mathcal{A}$  to  $X = \alpha_1^\perp$  is

$$\mathcal{A}^{\alpha_1^\perp} = \{\beta'^\perp \mid \beta \in R, \beta \neq \alpha_1\},$$

where if  $\beta = \sum_{\alpha \in \Delta^K} b_\alpha \alpha$ , then

$$\beta' = \frac{1}{\gcd(b_\alpha \mid \alpha_1 \neq \alpha \in \Delta^K)} \sum_{\alpha_1 \neq \alpha \in \Delta^K} b_\alpha \alpha.$$

Thus we obtain the restriction by deleting the coordinate to  $\alpha_1$  and reducing to the shortest vector in the lattice.

*Example 3.4.* Let  $\alpha_1 = (1, 0, 0)$ ,  $\alpha_2 = (0, 1, 0)$ ,  $\alpha_3 = (0, 0, 1)$ .

- a) Let  $R_+^a := \{\alpha_1, \alpha_2, \alpha_3, (0, 1, 1), (1, 1, 0), (1, 1, 1)\}$ , and let  $\mathcal{A} := \{\alpha^\perp \mid \alpha \in R_+^a\}$  be the crystallographic arrangement of Lie type  $A_3$ . Then the restriction  $\mathcal{A}^H$  of  $\mathcal{A}$  to any hyperplane  $H = \alpha_i^\perp$  is the crystallographic arrangement of type  $A_2$  defined by  $\bar{R}_+^a = \{(1, 0), (0, 1), (1, 1)\}$ .
- b) Let  $R_+^a := \{\alpha_1, \alpha_2, \alpha_3, (0, 1, 1), (0, 1, 2), (1, 0, 1), (1, 1, 1), (1, 1, 2)\}$ , and let  $\mathcal{A} := \{\alpha^\perp \mid \alpha \in R_+^a\}$  (which is not of Lie type). Then the restriction  $\mathcal{A}^H$  of  $\mathcal{A}$  to the hyperplane  $H = \alpha_1^\perp$  is the crystallographic arrangement of Lie type  $B_2$  (or  $C_2$  depending on the used definition) defined by  $\bar{R}_+^a = \{(1, 0), (0, 1), (1, 1), (1, 2)\}$ .
- c) Now let

$$R_+^a = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (0, 1, 2), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 2)\},$$

$\mathcal{A} := \{\alpha^\perp \mid \alpha \in R_+^a\}$ , thus  $R^a$  is a root system of type  $B_3$ . Choose  $H := (1, 0, 0)^\perp$ . Then  $\mathcal{A}^H$  is the set of kernels of

$$\bar{R}_+^a = \{(0, 1), (1, 0), (1, 1), (1, 2)\}.$$

Looking more closely, we notice that there are several hyperplanes in  $\mathcal{A}$  which restrict to the same element of  $\mathcal{A}^H$ . For example, three different roots map to the root  $(1, 1)$ , namely  $(0, 1, 1), (1, 1, 1), (1, 2, 2)$ .

We can see two effects at this last example:

- (1) It may be useful to keep track of the number of hyperplanes falling together under restriction. Thus instead of considering arrangements of hyperplanes, one should consider arrangements of hyperplanes with multiplicities (these are called *multiarrangements*).
- (2) A-priori, the vectors with the deleted coordinate do not form a reduced root system, i.e. they may have the form  $k\alpha$  for  $k \in \mathbb{Z} \setminus \{\pm 1\}$  and  $\alpha$  in the lattice. The reduction requires to rescale a root, for example  $(1, 2, 2) \mapsto \frac{1}{2}(2, 2)$ .

It would make sense to include the information of these two situations into the setting of Cartan graphs, but this would possibly make things more complicated than necessary. Notice that in the connected simply-connected case, crystallographic arrangements, Weyl groupoids, or Cartan graphs are all uniquely determined by the roots at a single object, hence by a single set  $R_+^a \subseteq \mathbb{Z}^r$ .

*Definition 3.5* (compare [CH15]). We will say that a finite set  $\Phi \subseteq \mathbb{Z}^r$  is a *reduced root set of rank r* if there exists a Cartan graph  $\mathcal{C}$  of rank  $r$  and an injective linear map  $w : \mathbb{Z}^r \rightarrow \mathbb{Z}^r$  such that  $w((R^a)^a) = \Phi$  for some object  $a$ .

*Definition 3.6.* A *(nonreduced) root set of rank r* is a set  $\Phi \subseteq \mathbb{Z}^r$  such that

$$\left\{ \frac{1}{\gcd(a_1, \dots, a_r)} \alpha \mid \alpha = (a_1, \dots, a_r) \in \Phi \right\}$$

is a reduced root set.

A *root multiset* is a root set  $\Phi$  together with a map  $\Phi \rightarrow \mathbb{N}$ ,  $\alpha \mapsto m_\alpha$ .

**3.2. All parabolic restrictions of finite root systems.** By Theorem 2.13 the root systems are in  $1 : 1$  correspondence with crystallographic arrangements. Thus Theorem 2.14 is a complete classification of simply-connected Cartan graphs.

In the following we compute all restrictions to existing hyperplanes (parabolic restrictions, see Definition 3.2) of all root systems in terms of their arrangement:

**Theorem 3.7.** *In rank three, the only crystallographic arrangements which are parabolic restrictions from higher dimensional crystallographic arrangements are those of the infinite series  $A_3, B_3, C_3, \mathcal{A}_3^1(2), \mathcal{A}_3^2(2)$  and those sporadic arrangements labeled*

$$1, 2, 3, 6, 7, 8, 9, 13, 14, 15, 20, 23$$

*in [CH15]. From rank four to rank eight, all crystallographic arrangements except the sporadic reflection arrangements  $E_6, E_7$ , and  $E_8$  are restrictions of higher dimensional reflection arrangements.*

*Every crystallographic arrangement of rank greater than 8 is restriction of a reflection arrangement of an infinite series (i.e.  $A_n, B_n, C_n, D_n$ ).*

*Proof.* The first two assertion are straightforward calculation performed by the computer using the classification, see Theorem 2.14. The last assertion is proven as follows: Every crystallographic arrangement in dimension greater than 8 is either a reflection arrangement (Weyl arrangement) or an arrangement denoted  $\mathcal{A}_\ell^k(2)$  in [OT92, 6]. But according to [OT92, Table 6.2],  $\mathcal{A}_\ell^k(2)$  is the restriction of an arrangement  $\mathcal{A}_{\ell'}^0$  for some  $\ell'$  large enough, and  $\mathcal{A}_{\ell'}^0$  is the arrangement of type  $D_{\ell'}$ .  $\square$

**3.3. Folding restriction.** A second important source of examples is as follows: We say that a finite group  $G$  acts on an arrangement  $\mathcal{A}$  if  $G$  acts on  $V$  such that  $g.\mathcal{A} = \mathcal{A}$  for any  $g \in G$ . Moreover we say  $G$  acts on a crystallographic arrangement  $(\mathcal{A}, R)$  if  $g.R = R$  for all  $g \in G$ , where  $g$  acts on  $V^*$  by  $(g.\alpha)(v) = \alpha(g^{-1}(v))$  for  $\alpha \in V^*$ .

If there exists a chamber  $K$  where  $G$  permutes the simple roots, we call the action a *permutation action* and it is equivalent to a permutation action of  $G$  on the Dynkin diagram resp. Cartan matrix of  $R$  in  $K$ .

**Definition 3.8.** A *folding restriction* of an arrangement  $\mathcal{A}$  with an action of  $G$  is the restriction  $\mathcal{A}^X$  to the subspace  $X = V^G$  of fixed points. We similarly define a folding restriction and a permutation restriction of a crystallographic arrangement  $(\mathcal{A}, R)$ .

In the sequel we will use the following notation. If  $(\mathcal{A}, R)$  is crystallographic and  $K$  is a chamber of  $\mathcal{A}$ , then

$$R_+^K := \left\{ \alpha \in R \mid \alpha \in \sum_{\gamma \in \Delta^K} \mathbb{N}_0 \gamma \right\}.$$

Thus  $R = R_+^K \dot{\cup} -R_+^K$  for every  $K$ . The sets  $R_+^K$  should not be confused with the positive roots  $R_+^a$  at an object of a Weyl groupoid. If  $a$  is the object corresponding to the chamber  $K$ , then  $R^a \subseteq \mathbb{Z}^r$  is the set of coordinate vectors with respect to  $\Delta^K$ .

We show the following easy characterization which generalizes the approach in [Ar62] for Lie type arrangements to classify real Lie algebras:

**Lemma 3.9.** *Let  $g$  be an involutive automorphism of the crystallographic arrangement  $(\mathcal{A}, R)$ . Then the folding restriction can be decomposed into two steps: First a parabolic restriction of  $\mathcal{A}$  to  $X_1 \in L(\mathcal{A})$  where  $X_1$  is a suitable subspace invariant under  $g$ , then by a permutation restriction of  $\mathcal{A}^{X_1}$  with respect to a suitable chamber  $K_2$ .*

*Proof.* The proof yields  $X_1, a_2$  and the explicit permutation action, and provides an efficient diagrammatic description of the possible actions of cyclic groups on crystallographic arrangements (comparable to Satake diagrams in [Ar62]):

- a) Let  $K$  be a chamber, such that  $|g.R_+^K \cap -R_+^K|$  is minimal, which surely exists. Denote  $\Delta = \Delta^K$  the positive simple roots at  $K$ .
- b) If  $\alpha \in \Delta$ , then  $g.\alpha \in R_+^K$  or  $g.\alpha = -\alpha$ : Otherwise, consider the chamber  $K'$  adjacent to  $K$  with  $R_+^{K'} = (R_+^K \setminus \{\alpha\}) \cup \{-\alpha\}$ . If  $g.\alpha = -\beta \in -R_+^K$  and  $\beta \neq \alpha$ , then

$$|g.R_+^{K'} \cap -R_+^{K'}| = |g.R_+^K \cap -R_+^K| - 1,$$

which contradicts the assumed minimality. Note that we use here that  $g$  is involutive, such that  $g.\alpha = -\beta$  also implies  $g^{-1}.\alpha = -\beta$ , so no root in  $R_+^K$  maps to  $\alpha$ .

- c) Let  $\Delta_1 := (-g^{-1}.\Delta) \cap \Delta$ , thus the set of  $\alpha \in \Delta$  with  $g.\alpha = -\alpha$ . Define  $X_1 := \Delta_1^\perp \in L(\mathcal{A})$ . Since  $g.\alpha = -\alpha$  for  $\alpha \in \Delta_1$ ,  $X_1$  is invariant under  $g$ . Further, if  $v \in X = V^G$ , then  $g.v = v$ . Hence  $\alpha(v) = (-\alpha)(v)$  for all  $\alpha \in \Delta_1$ , i.e.  $X \subseteq X_1$ .
- d) Consider the partition  $\Delta := \Delta_1 \cup \Delta_2$ , in particular  $g.\Delta_2 \subset R_+^K$ . We claim that there is a permutation  $g(\_)$  of the set  $\Delta_2$ , such that  $g.\alpha_i = \alpha_{g(i)} + \mathbb{N}\Delta_1$ . Indeed, modulo  $\mathbb{R}\Delta_1$  the actions of  $g, g^{-1}$  are inverse positive integer matrices, hence permutation matrices.

- e) Consider the parabolic restriction  $\mathcal{A}^{X_1}$  and let  $K_2 := K \cap X_1$  be the respective chamber, then by the above,  $g$  acts as a permutation on the crystallographic arrangement  $\mathcal{A}^{X_1}$  and the full restriction to  $X = V^G \subseteq X_1$  is hence a permutation restriction.

□

*Remark 3.10.* Even if  $(\mathcal{A}^X, \bar{R})$  is not crystallographic, it could be crystallographic with respect to a different choice of roots.

It is *not* true that every parabolic restriction can be obtained from a folding restriction for some suitable automorphism  $g$ :

**Lemma 3.11.** *For  $J \subset I$ , there exists an automorphism  $g$ , such that parabolic restriction to  $X = J^\perp$  coincides with folding restriction by  $g$ , if and only if the permutation automorphism (i.e. diagram automorphism)  $f_J := -w_J$  (on the parabolic subsystem generated by  $J$  with  $w_J$  the longest element), together with the identity permutation on all simple roots  $I \setminus J$  is a permutation automorphism for the entire arrangement.*

Take as counterexample  $A_2 \subset A_3$  or also  $A_2 \subset A_4$ . The condition is however always fulfilled for parabolic restrictions to one hyperplane  $H \in \mathcal{A}$ .

*Proof.* Let  $f$  be such an extension of  $f_J = -w_J$  and consider the automorphism  $g := w_J f$ . On the subsystem  $R_J^K$  it acts as  $-\text{id}$ , while on the remaining positive roots it acts by adding terms in  $R_{J,+}^K$  i.e. as the identity modulo  $R_J^K$ . We now apply Lemma 3.9: We see that for a simple root  $\alpha_i \in \Delta^K$ , by construction we have  $g.\alpha \in R_+^K$  for  $i \in I \setminus J$  or  $g.\alpha_i = -\alpha_i$  for  $i \in J$ . Hence  $\Delta_1 = J$  and thus the first parabolic restriction  $X_1 = J^\perp$  is the parabolic restriction in question. Furthermore, since  $g$  acts on the remaining simple roots by identity modulo  $R_J^K$ , the second permutation restriction is trivial.

Vice-versa, for any automorphism  $g$  by Lemma 3.9 we have a chamber  $K$  and a set of simple roots  $\Delta_1 \subset \Delta$  with the property  $g.\alpha_i = -\alpha_i$ , and a permutation  $\pi$  on the other roots  $\Delta \setminus \Delta_1$  such that  $g.\alpha \in \pi(\alpha) + R_{+,J}^K$ . The folding restriction by  $g$  is a parabolic restriction to  $\Delta_1$  and then a permutation restriction.

Since it needs to coincide by assumption with parabolic restriction to  $J$ , we have  $J \subset \Delta$ , moreover  $J = \Delta$  and  $\sigma = \text{id}$ . We can consider  $w_J^{-1}g$ , which is a permutation automorphism  $f$ . On the subsystem  $J$ ,  $f$  is equal to  $-w_J =: f_J$  and on the remaining roots the identity, as claimed. □

*Remark 3.12.* Compare the condition of this Lemma to the condition fulfilled by the quotient root systems used for parabolic induction of cuspidal representations in Lusztig's character theory of finite Lie groups.

Note that the permutation induces a permutation automorphism of root systems already on the localization to  $\Delta_2$ , in particular a diagram automorphism of the sub-Dynkin diagram. One may hence enumerate all possible automorphism by listing root systems, where for some object  $a$ ,  $R^a$  has a symmetric sub-diagram (and say color all nodes in  $\Delta_1$  black) as done for Satake diagrams. Alternatively (and maybe more feasible for us) one may compute all parabolic restrictions as done below and then enumerate all full permutation automorphisms. A different way to classify root system automorphisms (even if  $g$  is *not involutive*) is

**Lemma 3.13.** *For any automorphism  $g$  of a crystallographic arrangement and any chamber  $K$  we can write  $g = wf$ , where  $w$  is the Weyl groupoid element defined by  $w(K) = g(K)$  and  $f$  is a permutation automorphism of the object  $K$  (so in particular a diagram automorphism of the respective Dynkin diagram, possibly trivial).*

*Proof.* This is easily proven as for Lie algebras: The autmorphism  $g$  has to send a chamber  $K$  to some chamber  $K'$ . By transitivity of the Weyl groupoid on the set of chambers we find a (unique)  $w$  with  $w(K) = K'$ . Then both  $w^{-1}g$  and  $g^{-1}w$  leave  $K$  invariant and act hence as mutually inverse, integral positive matrix on  $R_+^K$ . They are hence again permutation matrices.  $\square$

It is not as easy as for parabolic restriction to determine the resulting root system of a folding restriction. In the special case of permutation restriction, the new set of roots  $\bar{R}$  consists of the orbits of  $R$  under  $G$ :

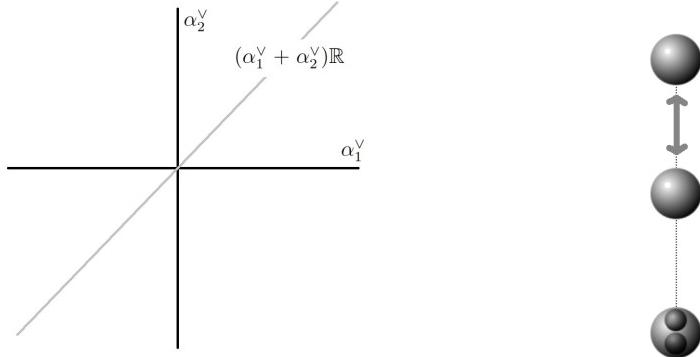
*Example 3.14.* In all of the following examples, the diagram automorphism inducing  $g$  is pictured, as well as the resulting root system including multiplicities. Note that foldings for Lie type such as a)-c) are of course well-known.

- a) Let  $R_+^a = \{(1, 0), (0, 1)\}$  be of Lie type  $A_1 \times A_1$  and  $g$  the permutation transposing  $\alpha_1 \leftrightarrow \alpha_2$ . A basis for  $V$  is  $\alpha_1^\vee, \alpha_2^\vee$ , hence the invariant subspace is  $X = v\mathbb{R} := (\alpha_1^\vee + \alpha_2^\vee)\mathbb{R}$ . The restriction of the arrangement is of course of type  $A_1$  (two of the four chambers are intersected). The restrictions of the roots  $\alpha_1, \alpha_2 \in R$  to  $(\mathcal{A}^X, \bar{R})$  are both  $\bar{\alpha}_1 := (v \mapsto 1)$  which can be expressed in

terms of the orbit  $G\alpha_1 = \{\alpha_1, \alpha_2\}$  as follows

$$\bar{\alpha}_1 = \frac{\alpha_1 + \alpha_2}{2} = \alpha_{\{1,2\}} \quad \text{where } \alpha_{G\alpha} := \frac{1}{|G|} \sum_{g \in G} g.\alpha$$

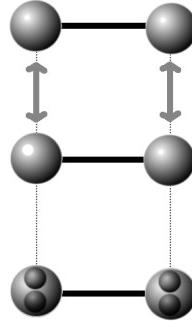
(this also justifies our initial choice of scaling  $v = \bar{\alpha}_1^\vee$ ). Hence  $\bar{R}_+^a$  is reduced of Lie type  $A_1$  with root  $\bar{\alpha}_1 = \alpha_{\{1,2\}}$ . The root  $\bar{\alpha}_1$  has *multiplicity* 2 since two roots  $\alpha_1, \alpha_2$  (resp. hyperplanes  $H_1, H_2$ ) restrict to the root  $\bar{\alpha}_1$  (resp. hyperplane); see Section 3.1.



- b) Let  $R_+^a = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}$  be of Lie type  $A_2 \times A_2$  and  $g$  the permutation transposing  $\alpha_1 \leftrightarrow \alpha_2, \alpha_3 \leftrightarrow \alpha_4$ . Then the restriction is  $\bar{R}_+^a = \{(1, 0), (0, 1), (1, 1)\}$  with two simple roots and one non-simple root each corresponding to an orbit of length 2:

$$\begin{aligned} \bar{\alpha}_1 &= \alpha_{\{1,2\}} \\ \bar{\alpha}_2 &= \alpha_{\{3,4\}} \\ \bar{\alpha}_1 + \bar{\alpha}_2 &= \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{2} = \alpha_{\{13,24\}} \end{aligned}$$

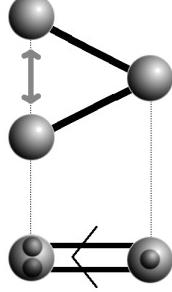
The restriction is hence a reduced root system of Lie type  $A_2$  and all roots have multiplicity 2.



- c) Let  $R_+^a = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1)\}$  be of Lie type  $A_3$  and  $g$  the permutation transposing  $\alpha_1 \leftrightarrow \alpha_3$ . Then the restriction is  $\bar{R}_+^a = \{(1, 0), (0, 1), (1, 1), (2, 1)\}$  with the following roots:

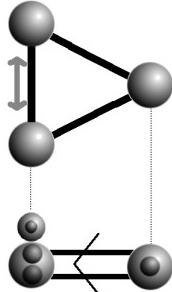
$$\begin{aligned}\bar{\alpha}_1 &= \alpha_{\{1,3\}} \\ \bar{\alpha}_2 &= \alpha_2 \\ \bar{\alpha}_1 + \bar{\alpha}_2 &= \frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} = \alpha_{\{12,23\}} \\ 2\bar{\alpha}_1 + \bar{\alpha}_2 &= \alpha_1 + \alpha_2 + \alpha_3 = \alpha_{123}\end{aligned}$$

The restriction is hence reduced of Lie type  $B_2$  with short roots of multiplicity 2 and long roots of multiplicity 1.



- d) Let  $R_+^a = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$ , which is *not* of Lie type, let  $g$  be the permutation transposing  $\alpha_1 \leftrightarrow \alpha_3$ . Then the restriction is  $\bar{R}_+^a = \{(1, 0), (0, 1), (1, 1), (2, 0), (2, 1)\}$  and hence of Lie type  $B_2$  and multiplicities 1, 2 as in the previous case, but this time non-reduced, i.e.

$$[M_{\bar{\alpha}_1}, M_{\bar{\alpha}_1}] = M_{2\bar{\alpha}_1} \neq 0$$



A similar effect appears already when folding a so-called loop  $A_2$ , see [Len14a].

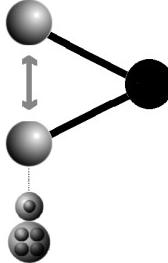
We finally give examples for general folding restrictions.

*Example 3.15.* Similarly to Satake diagram (first example), according to the proof of Lemma 3.9, we draw the permutation of the simple roots in  $\Delta_2$  and denote the simple roots in  $\Delta_1$  by blackend dots.

Let  $R_+^K = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1)\}$  be of Lie type  $A_3$  and  $g$  the automorphism

$$g := \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have  $g.R_+^K \cap -R_+^K = \{-\alpha_2\}$  which is minimal (so  $K$  is already chosen suitably) and thus  $\Delta_1 = \{\alpha_2\}$  and  $\Delta_2 = \{\alpha_1, \alpha_3\}$ . Moreover, modulo  $\Delta_1$  we have a permutation  $g : \alpha_1 \leftrightarrow \alpha_2$ . The restriction to  $X_1 = \alpha_2^\perp$  is of type  $A_2$  with multiplicities 2 for the two simple roots and 1 for the non-simple root. Note that while the root system is of Lie type, the multiplicities are not invariant under all reflections and we thus get a Weyl groupoid covering the Weyl group  $A_3$  (similar to a Lie superalgebra). As a second step, the permutation restriction by the transposition  $g$  yields a non-reduced root system of Lie type  $A_1$  where  $\bar{\alpha}_1$  has multiplicity 4 and  $2\bar{\alpha}_1$  has multiplicity 1.



#### 4. ARRANGEMENTS AND NICHOLS ALGEBRAS

**4.1. Nichols algebras.** Let  $\mathcal{C}$  be a braided category, then there is a straightforward notion of a Hopf algebra in  $\mathcal{C}$ . For this and especially the notion of Yetter-Drinfel'd modules in braided categories see e.g. [BLS14]. To simplify the discussion we assume in this article  $\mathcal{C} = {}_h^h\mathcal{YD}$  where  $h$  is a complex Hopf algebra.

A particularly well-known example class are the diagonal Nichols algebras. In Section 4.4 the reader can find a table comparing the notation for general Nichols algebras with the notation for quantum groups and general diagonal Nichols algebras in [Lusz94] and [Heck07].

*Definition 4.1.* Let  $M = M_1 \oplus \cdots \oplus M_n$  be a semisimple object in  $\mathcal{C}$  with simple summands  $M_i$ . We call  $n$  the *rank* of  $M$ . The *Nichols algebra*  $\mathcal{B}(M)$  is an  $\mathbb{N}$ -graded Hopf algebra in the category  $\mathcal{C}$  with  $\mathcal{B}(M)_0 = 1\mathbb{C}$  the identity object in  $\mathcal{C}$  and  $\mathcal{B}(M)_1 = M$  primitive elements  $\Delta(m) = 1 \otimes m + m \otimes 1$  which generate  $\mathcal{B}(M)$  as an algebra.

The Nichols algebra  $\mathcal{B}(M)$  can be defined by the following equivalent properties:

- Every  $\mathbb{N}$ -graded Hopf algebra  $B'$  with the above properties admits a unique graded Hopf algebra surjection  $B' \rightarrow \mathcal{B}(M)$  which is the identity on  $M$ . That is,  $\mathcal{B}(M)$  is the quotient of the tensor algebra  $T(M)$  by the largest Hopf ideal  $\mathfrak{J}$  in degree  $\geq 2$ .
- The pairing  $M \otimes M^* \rightarrow \mathbb{C}$  induces a Hopf pairing  $\mathcal{B}(M) \otimes \mathcal{B}(M^*) \rightarrow \mathbb{C}$  which is nondegenerate. That is,  $\mathcal{B}(M)$  is the quotient of the tensor algebra  $T(M)$  by the radical of the Hopf pairing  $T(M) \otimes T(M^*) \rightarrow \mathbb{C}$ .
- More explicitly,  $\mathcal{B}(M)$  is the quotient of  $T(M)$  in each degree  $k$  by the kernel of the *quantum symmetrizer*, which is defined in terms of the braid group action  $M^{\otimes k} \rightarrow M^{\otimes k}$ .

While the last characterization enables one in principle to compute  $\mathcal{B}(M)$  in each degree, it is extremely difficult to find generators and relations for  $\mathcal{B}(M)$  or even determine for a given  $M$  if it is finite-dimensional.

Since  $\mathcal{B}(M)$  is an  $\mathbb{N}$ -graded algebra with finite-dimensional degree layers, we may consider the *Hilbert series*, a formal power series

$$\mathcal{H}(t) := \sum_{k \geq 0} \dim(\mathcal{B}(M)_k) t^k.$$

In particular, if the dimension is finite then  $\dim(\mathcal{B}(M)) = \mathcal{H}(1)$ . We frequently use the symbol  $(n)_t := 1 + t + t^2 + \cdots + t^{n-1}$ .

The first examples serves to fix notation:

*Example 4.2* (Diagonal case). Let  $h = \mathbb{C}[\Gamma]$  and  $\Gamma$  be an abelian group. Then the braided category  $\mathcal{C} = {}^h\mathcal{YD}$  is semisimple. The finite-dimensional simple objects are 1-dimensional vector spaces  $M_i = x_i\mathbb{C}$  together with a group element  $g_i \in \Gamma$  and a linear character  $\chi_i : G \rightarrow \mathbb{C}^\times$ . The braiding is given by

$$\begin{aligned} M_i \otimes M_j &\rightarrow M_j \otimes M_i, \\ x_i \otimes x_j &\mapsto q_{ij} x_j \otimes x_i, \quad q_{ij} := \chi_j(g_i). \end{aligned}$$

We call such a braiding *diagonal* and  $q_{ij}$  the *braiding matrix*. From the third characterization we see that the Nichols algebra  $\mathcal{B}(M)$  depends only on the braiding matrix  $(q_{ij})_{i,j}$  of  $M$ . Conversely, every diagonal braiding can be realized as Yetter-Drinfel'd module, say over  $\Gamma = \mathbb{Z}^n$ .

*Example 4.3.* Let  $q \in \mathbb{C}^\times$ . Let  $M = M_1$  be the one-dimensional complex braided vector space with basis  $x_1$  and braiding matrix  $(q_{11}) = (q)$ . The Nichols algebra is

$$\mathcal{B}(M) \cong \mathbb{C}[x_1]/(x_1^\ell)$$

if  $q$  is a primitive  $\ell$ -th root of unity, and  $\mathcal{B}(M) \cong \mathbb{C}[x_1]$  if  $q = 1$  or not a root of unity. In the first case, the Hilbert series is  $(\ell)_t$ , in the second case  $\frac{1}{1-t}$ .

The next example exhibits the role of Nichols algebras as quantum Borel parts.

*Example 4.4.* Let  $q$  be a primitive  $\ell$ -th root of unity and assume for simplification  $2, 3 \nmid \ell$  and  $\ell > 3$ . Let  $\mathfrak{g}$  be a complex finite-dimensional semisimple Lie algebra of rank  $n$ , let  $\alpha_1, \dots, \alpha_n$  be a set of simple roots,  $R_+$  the set of positive roots, and let  $(,)$  be the Killing form, normalized to  $(\alpha_i, \alpha_i) = 2$  for short simple roots  $\alpha_i$ .

Consider the Yetter-Drinfel'd modules  $M = M_1 \oplus \dots \oplus M_n$  over the abelian group  $\Gamma = \mathbb{Z}_\ell^n$  generated by  $g_i$  and  $M_i = E_i \mathbb{C}$  with group element  $g_i$  and character  $\chi_i(g_j) := q^{(\alpha_i, \alpha_j)}$ .

Then  $\mathcal{B}(M)$  is finite-dimensional with Hilbert series  $\prod_{\alpha \in R_+} (\ell)_{t^{\deg(\alpha)}}$ , thus of dimension  $\ell^{|R_+|}$ . In fact  $\mathcal{B}(M) \cong u_q(\mathfrak{g})^+$  is as an algebra the positive part of the small quantum group.

**4.2. The Weyl groupoid of a Nichols algebra.** The structure theory of Nichols algebras is dominated by the structure of the Weyl groupoid, which generalizes the role of reflection operators in quantum groups as introduced by Lusztig [Lusz90], and allows to define a root system for the Nichols algebra. For  $M$ ,  $q_{ij}$  diagonally braided (see above) Heckenberger has introduced in a series of papers reflections [Heck07] and an arithmetic root system in terms of the bicharacter induced by  $q_{ij}$  and finally classified all finite-dimensional Nichols algebras in terms of their root systems in [Heck09]. More generally in [AHS10][HS10] Andruskiewitsch, Heckenberger, and Schneider have introduced a Weyl groupoid and root system for arbitrary semisimple  $M \in {}^h\mathcal{YD}$ . We shall sketch their approach in the form discussed in [HS13] or [BLS14]:

Let  $M = M_1 \oplus \dots \oplus M_n$  be a finite-dimensional semisimple object in  $\mathcal{C} = {}^h\mathcal{YD}$  and assume for simplicity that  $\mathcal{B}(M)$  is already finite-dimensional. For any  $i$  the projection  $M \rightarrow M_i$  induces a projection  $\pi_{\mathcal{B}}(M) \rightarrow \mathcal{B}(M_i)$ . By the *Radford projection theorem* we may write

$$\mathcal{B}(M) \cong K_i \rtimes \mathcal{B}(M_i), \quad K_i := \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_i)} := \{h \mid (\text{id} \otimes \pi_i)\Delta(h) = h \otimes 1\}$$

where the space of coinvariants  $K_i$  with respect to  $\pi_i$  is a Hopf algebra in the braided category  ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})$  and  $\rtimes$  is the *Radford biproduct*.

By the second characterization of Nichols algebras we have a non-degenerate Hopf pairing  $\mathcal{B}(M) \otimes \mathcal{B}(M^*) \rightarrow \mathbb{C}$ , in particular  $\dim(\mathcal{B}(M)) = \dim(\mathcal{B}(M^*))$ . One can show that this Hopf pairing induces a category equivalence  $\Omega : {}^{\mathcal{B}(M)}_{\mathcal{B}(M)}\mathcal{YD} \rightarrow {}^{\mathcal{B}(M^*)}_{\mathcal{B}(M^*)}\mathcal{YD}$ . Now we may turn  $\Omega(K_i) \in {}^{\mathcal{B}(M^*)}_{\mathcal{B}(M^*)}\mathcal{YD}$  again into a Hopf algebra in  $\mathcal{C}$ , the *reflection*:

$$r_i(\mathcal{B}(M)) := \Omega(K_i) \rtimes \mathcal{B}(M^*).$$

Reflection does clearly not change the dimension  $\dim(\mathcal{B}(M)) = \dim(r_i(\mathcal{B}(M)))$ . Otherwise  $\mathcal{B}(M), r_i(\mathcal{B}(M))$  can be quite different, although one can prove  ${}^{\mathcal{B}(M)}_{\mathcal{B}(M)}\mathcal{YD} \cong {}^{r_i(\mathcal{B}(M))}_{r_i(\mathcal{B}(M))}\mathcal{YD}$ . The reflection operation including the mentioned properties are neither restricted to Nichols algebras nor to the category  $\mathcal{C} = {}^h\mathcal{YD}$  and were dubbed *partial dualization* in [BLS14].

For a Nichols algebra  $\mathcal{B}(M)$ , Andruskiewitsch, Heckenberger and Schneider describe this operation in much more detail: In particular,  $r_i(\mathcal{B}(M)) \cong \mathcal{B}(R_i(M))$  is again a Nichols algebra for some explicit  $R_i(M) \in \mathcal{C}$ . We summarize some results of [AHS10], [HS10], and [HS13]:

**Theorem 4.5.** *Let  $h$  be a complex Hopf algebra. Let  $M_i$  be a finite collection of simple  $h$ -Yetter-Drinfel'd modules. Consider  $M := \bigoplus_{i=1}^n M_i \in {}^h\mathcal{YD}$  and assume that the associated Nichols algebra  $H := \mathcal{B}(M)$  is finite-dimensional. Then the following assertions hold:*

- *By construction, the Nichols algebras  $\mathcal{B}(M), r_i(\mathcal{B}(M))$  have the same dimension as complex vector spaces.*
- *For  $i \in I$ , denote by  $\hat{M}_i$  the braided subspace*

$$\hat{M}_i = M_1 \oplus \dots \oplus M_{i-1} \oplus M_{i+1} \oplus \dots \oplus M_n$$

of  $M$ . Denote by  $\text{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$  the braided vector space obtained as the image of  $\hat{M}_i \subset \mathcal{B}(M)$  under the adjoint action of the Hopf subalgebra  $\mathcal{B}(M_i) \subset \mathcal{B}(M)$ . Then, there is a unique isomorphism [HS13, Prop. 8.6]

$$K_i \cong \mathcal{B}(\text{ad}_{\mathcal{B}(M_i)}(\hat{M}_i))$$

of Hopf algebras in the braided category  $\overset{\mathcal{B}(M_i)}{\mathcal{B}(M_i)}\mathcal{YD}$  ( ${}^h\mathcal{YD}$ ) which is the identity on  $\text{ad}_{\mathcal{B}(M_i)}(\hat{M}_i)$ .

- Define, with the usual convention for the sign,

$$-c_{ij} := \max\{m \mid \text{ad}_{M_i}^m(M_j) \neq 0\} \quad c_{ii} = 2.$$

Fix  $i \in I$  and denote for  $j \neq i$

$$V_j := \text{ad}_{M_i}^{-c_{ij}}(M_j) \subset \mathcal{B}(M).$$

The braided vector space

$$R_i(M) = V_1 \oplus \cdots M_i^* \cdots \oplus V_n \in {}^h\mathcal{YD}$$

is called the the  $i$ -th reflection of the braided vector space  $M$ . Then there is a unique isomorphism [HS13, Thm. 8.9] of Hopf algebras in  ${}^h\mathcal{YD}$

$$r_i(\mathcal{B}(M_1 \oplus \cdots \oplus M_n)) \cong \mathcal{B}(V_1 \oplus \cdots M_i^* \cdots \oplus V_n)$$

which is the identity on  $M$ .

- With the same definition for  $c_{ij}$  for  $i \neq j$  and  $c_{ii} := 2$ , the matrix  $(c_{ij})_{i,j=1,\dots,n}$  is a generalized Cartan matrix [AHS10, Thm. 3.12]. Moreover, one has  $r_i^2(\mathcal{B}(M)) \cong \mathcal{B}(M)$ , and the Cartan matrices coincide,  $c_{ij}^M = c_{ij}^{r_i(M)}$ . One obtains a Cartan graph where each object  $a \in A$  corresponds to some reflection  $\mathcal{B}(M^a)$ .
- The maps  $r_i$  give rise to a Weyl groupoid: The objects are the different isomorphism classes of tuples  $(M_1, \dots, M_n)$  and the formal morphisms  $M \rightarrow r_i(M)$  are generated by reflections  $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$  with respect to  $c_{ij}^M$ .
- Let  $R_+^a$  the set of positive roots, and choose a reduced expression for the longest element in the Weyl groupoid, then there is an isomorphism of  $\mathbb{N}$ -graded objects in  $\mathcal{C}$  (not algebras), or PBW-basis:

$$\mathcal{B}(M^a) = \bigotimes_{\alpha \in R_+^a} M_\alpha.$$

Here  $M_\alpha$  are certain simple object in  $\mathcal{C}$ , namely  $= M_i^a$  if  $H_\alpha$  is adjacent to the object  $a$ . For details, we refer to [AHS10, Sect. 3.5] and [HS10, Sect. 5].

Note that a crucial point is that the algebras  $\mathcal{B}(M), r_i(\mathcal{B}(M))$  can be quite different and in particular their Cartan matrix  $(c_{ij})_{ij}$  may be different – in contrast to quantum groups, where all reflections are isomorphic as algebras. This is why we obtain a Weyl groupoid instead of a Weyl group, a Cartan graph instead of a single Cartan matrix, and more root systems than for semisimple Lie algebras.

**Theorem 4.6** (Diagonal case, [Heck09]). *For a diagonally braided vector space  $M, q_{ij}$  Heckenberger had previously described the reflection  $r_i(M), q'_{ij}$  as the explicit base transformation of the bicharacter induced by the reflection  $\alpha_j \mapsto \alpha_j - c_{ij}\alpha_i, \alpha_i \mapsto -\alpha_i$ , where the Cartan matrix  $(c_{ij})_{ij}$  associated to  $(M, (q_{ij})_{i,j})$  can be calculated by*

$$c_{ii} = 2, \quad c_{ij} = -\min\{m \mid (m+1)_{q_{ii}} = 0 \text{ or } q_{ii}^m q_{ij} q_{ji} = 1\}, \quad i \neq j.$$

*Any finite-dimensional complex Nichols algebra in the category of Yetter-Drinfel'd modules over an abelian group  $\Gamma^{\text{YD}}$  appears in Heckenberger's list [Heck09].*

The respective root systems are many, but not all possible root systems and Weyl groupoids as classified by Heckenberger and the first author, see Section 2.4. For instance, there are infinitely many Weyl groupoids of rank 2 and many more exceptional Weyl groupoids in rank 3.

The following class of examples of type  $B_2$  contains as special case the Borel part  $u_q(B_2)^+$  when the braiding matrix  $(q_{ij})_{ij}$  is chosen symmetric.

*Example 4.7.* Let  $q, q^4 \neq 1$  be an  $\ell$ -th root of unity and  $M_{\alpha_1}, M_{\alpha_2}$  be 1-dimensional Yetter-Drinfel'd modules (say over  $\Gamma = \mathbb{Z}^2$ ) such that the diagonal braiding of  $M^a := M_{\alpha_1} \oplus M_{\alpha_2}$  fulfills

$$q_{\alpha_1 \alpha_1} = q^2, \quad q_{\alpha_i \alpha_j} q_{\alpha_j \alpha_i} = q^{-4}, \quad q_{\alpha_2 \alpha_2} = q^4.$$

The previous formula yields the Cartan matrix and Dynkin diagram  $M^a$ :

$$(c_{ij}^a)_{i,j} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \text{Dynkin diagram: } \begin{array}{c} \bullet < \bullet \end{array}$$

The reflection on the short root  $\alpha_1$  given by this Cartan matrix gives for  $r_1(M)$   $\alpha'_1 = -\alpha_1$  and  $\alpha'_2 = \alpha_2 + 2\alpha_1$ . For the braiding matrix of  $r_1(M)$  we thus get by

base transformation of the bicharacter:

$$\begin{aligned} q_{\alpha'_1 \alpha'_1} &= q_{\alpha_1 \alpha_1} = q^2 \\ q_{\alpha'_2 \alpha'_2} &= q_{\alpha_2 \alpha_2} q_{\alpha_1 \alpha_2}^2 q_{\alpha_2 \alpha_1}^2 q_{\alpha_1 \alpha_1}^4 = q^{4-8+8} = q^4 \\ q_{\alpha'_1 \alpha'_2} &= q_{\alpha_1 \alpha_2}^{-1} q_{\alpha_1 \alpha_1}^{-2} \\ q_{\alpha'_2 \alpha'_1} &= q_{\alpha_1 \alpha_2}^{-1} q_{\alpha_1 \alpha_1}^{-2} \end{aligned}$$

We observe that in general the Yetter-Drinfel'd module  $r_1(M)$  given by  $q_{\alpha'_i \alpha'_j}$  is *not* isomorphic to  $M$  (for the special case  $q_{ij} = q_{ji}$  appearing in  $u_q(B_2)^+$  it is!). However we easily check

$$q'_{\alpha_1 \alpha_2} q'_{\alpha_2 \alpha_1} = q_{\alpha_1 \alpha_2}^{-1} q_{\alpha_2 \alpha_1}^{-1} q_{\alpha_1 \alpha_1}^{-4} = q^{4-8} = q_{\alpha_i \alpha_j} q_{\alpha_j \alpha_i},$$

so  $r_1(M) = M_{\alpha'_1} \oplus M_{\alpha'_2}$  has the same Cartan matrix as  $M$  (this can not be deduced merely from the Dynkin diagram). The same can be checked for  $r_2(M)$ . We thus obtain a Cartan graph with a single Cartan matrix for all reflections  $M^a$ . The arrangement and set of positive roots is of Lie type  $B_2$ :

$$R_+^a = \{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\} = \{(1, 0), (0, 1), (1, 1), (2, 1)\}$$

with notation  $\alpha_{12} := \alpha_1 + \alpha_2$ ,  $\alpha_{112} := 2\alpha_1 + \alpha_2$ . The self-braidings are

$$q_{\alpha_1 \alpha_1} = q_{\alpha_{12} \alpha_{12}} = q^2, \quad q_{\alpha_2 \alpha_2} = q_{\alpha_{112} \alpha_{112}} = q^4.$$

Hence the PBW-basis Theorem gives an isomorphism of  $\mathbb{N}$ -graded  $\Gamma$ -Yetter-Drinfel'd modules:

$$\begin{aligned} \mathcal{B}(M^a) &\cong \bigotimes_{\alpha \in R_+^a} M_\alpha \\ &= \mathbb{C}[t_1]/(t_1^{\text{ord}(q^2)}) \otimes \mathbb{C}[t_2]/(t_2^{\text{ord}(q^4)}) \otimes \mathbb{C}[t_{12}]/(t_{12}^{\text{ord}(q^2)}) \otimes \mathbb{C}[t_{112}]/(t_{112}^{\text{ord}(q^4)}). \end{aligned}$$

For  $\ell$  odd,  $q^2, q^4$  both have order  $\ell$ , hence the Hilbert series is  $(\ell)_t^2 (\ell)_{t^2} (\ell)_{t^3}$  and in particular the dimension is  $\ell^4$ .

We also give an example which is not of Lie type and which we will use in the following. It is a finite-dimensional Nichols algebra  $\mathcal{B}(M)$  of diagonal type of rank 3 appearing in [Heck09, row 11]. The rows 9, 10 define the same arrangement, but different roots of unity involved and hence more types of objects with the same Cartan matrix. The arrangement associated to this Nichols algebra as well as all restrictions are calculated in Section 5.3.

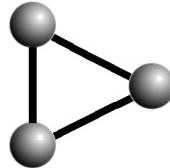
*Example 4.8.* Let  $M_{\alpha_1}, M_{\alpha_2}, M_{\alpha_3}$  be 1-dimensional Yetter-Drinfel'd modules (say over  $\mathbb{Z}^3$ ) such that the diagonal braiding of  $M^a := M_{\alpha_1} \oplus M_{\alpha_2} \oplus M_{\alpha_3}$  fulfills

$$q_{\alpha_i \alpha_i} = -1, \quad q_{\alpha_i \alpha_j} q_{\alpha_j \alpha_i} = \zeta,$$

with  $i \neq j$  and  $\zeta$  a primitive third root of unity. The associated crystallographic arrangement has 7 roots. It is called  $\mathcal{A}_3^1(2)$  and the first member of a series.

$$R_+^a = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}.$$

From this we easily see the Dynkin diagram/Cartan matrix is a simply-laced triangle and not of Lie type:



The self-braidings of these roots are

$$q_{\alpha_i \alpha_i} = -1, \quad q_{\alpha_i j, \alpha_i j} = \zeta, \quad q_{\alpha_1 \alpha_2} = -1.$$

Hence the Nichols algebra  $\mathcal{B}(M^a)$ ,  $M^a = M_{\alpha_1} \oplus M_{\alpha_2} \oplus M_{\alpha_3}$  has Hilbert series  $(2)_t^3 (3)_{t^2}^3 (2)_{t^3}$  and dimension 432.

We calculate the reflection to an object  $a'$  with respect to  $\alpha_2$ : The simple roots at  $a'$  are then  $\alpha'_1 := \alpha_{12}$ ,  $\alpha'_2 := -\alpha_2$ ,  $\alpha'_3 := \alpha_{23}$  and the braiding matrix fulfills

$$q_{\alpha'_1 \alpha'_1} = q_{\alpha'_3 \alpha'_3} = \zeta, \quad q_{\alpha'_2 \alpha'_2} = -1,$$

$$q_{\alpha'_1 \alpha'_2} q_{\alpha'_2 \alpha'_1} = q_{\alpha'_2 \alpha'_3} q_{\alpha'_3 \alpha'_2} = \zeta^{-1}, \quad q_{\alpha'_1 \alpha'_3} q_{\alpha'_3 \alpha'_1} = 1.$$

The respective roots in the basis  $\alpha'_1, \alpha'_2, \alpha'_3$  can easily be either again calculated or directly read off from transforming  $R_+^a$ :

$$R_+^{a'} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 2, 1)\}$$

which has Dynkin diagram/Cartan matrix  $A_2$ , but is not of Lie type, since it has a different root system.

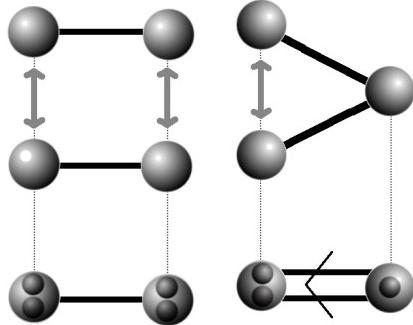


The Hilbert series is now  $(2)_t (3)_t^2 (2)_{t^2}^2 (2)_{t^3} (3)_{t^4}$  but of course the dimension is still 432.

We finally discuss nondiagonal Nichols algebras over nonabelian groups.

*Definition 4.9.* Let  $M$  be a Yetter-Drinfel'd module over the group ring of  $\mathbb{C}[G]$  of a nonabelian group  $G$ . Such an  $M$  decomposes into irreducible summands  $M_{\alpha_i} = M([g_i], \chi_i)$  associated to a conjugacy class  $[g_i] \subset G$  and an irreducible representation  $\chi$  of the centralizer of  $g_i$  (see e.g. [HS10]). In particular  $M = \bigoplus_i M_{\alpha_i}$  is a nondiagonally braided vector space and we study the Nichols algebra  $\mathcal{B}(M)$ .

The first finite-dimensional Nichols algebra of this type has been constructed first for the dihedral group  $G = \mathbb{D}_4$  in [MS00] in the context of Coxeter groups. Later, in [Len14a] it has been constructed by the second author as smallest example of a new family of large-rank finite-dimensional indecomposable Nichols algebras, introducing the folding construction of non-diagonal Nichols algebras from diagonal ones (in this case  $A_2 \times A_2$  with  $q = -1$ ). The reader may compare the root system below with the root system in 3.14 b). A similar construction for c) returns a decomposable Nichols algebra of type  $B_2$ .



*Example 4.10.* Let  $\Gamma = \mathbb{D}_4 = \langle a, b \rangle / (a^4 = b^2 = 1, ba = a^{-1}b)$  then one can define 2-dimensional simple Yetter-Drinfel'd modules  $M_{\alpha_1}, M_{\alpha_2}$  associated to the conjugacy classes  $[ab], [b]$ . In fact, the Nichols algebra  $\mathcal{B}(M)$  for  $M = M_{\alpha_1} \oplus M_{\alpha_2}$  has a root system of type  $A_2$  and  $M_{\alpha_{12}}$  is also 2-dimensional and associated to the conjugacy class  $[a]$ . Altogether this Nichols algebra has Hilbert series  $(2)_t^4(2)_t^2$  and thus dimension  $2^6$ , indeed just like  $u_q(A_2 \times A_2)^+$  with  $q = i$ .

Note that in [Heck09] all finite-dimensional complex Nichols algebras over abelian groups (equivalently with diagonal braiding) have been classified, and very recently in [HV14] all finite-dimensional complex indecomposable Nichols algebras over nonabelian groups. For later use we note that only a subset of all finite Weyl groupoids (Theorem 2.14) appear in these cases:

**Corollary 4.11.** *Using the labeling introduced in [CH15], the following crystallographic arrangements of rank three come from Weyl groupoids of finite-dimensional Nichols algebras of diagonal type<sup>6</sup>,*

*types  $A_3, B_3, C_3, \mathcal{A}_3^1(2), \mathcal{A}_3^2(2)$ , and the sporadic ones labeled 1, 2, 3, 6, 8, 9,*

*and the following come from complex finite-dimensional indecomposable Nichols algebras over nonabelian groups,*

*types  $A_3, C_3$ , and the sporadic one labeled 9.*

*Note that by Theorem 3.7, all of these are (parabolic) restrictions of Weyl arrangements.*

*Question 4.12.* As already mentioned in the previous example, all such Nichols algebras of rank  $< 4$  over nonabelian groups had been uniformly constructed in [Len14a] by *folding*; they have root systems of type  $A_n, C_n, E_6, E_7, E_8, F_4$ . The resulting root system was calculated by hand and can now be understood as a restriction. It would be interesting to give a larger class of constructions:

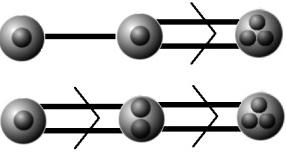
- By considering foldings of Nichols algebras of non-Lie type, such as Example 3.14 d).
- By considering Satake-type foldings with nontrivial  $\Delta_1$ , which tends to drastically decrease the rank. This leads a-priori to Nichols algebras in other braided categories.
- In the affine setting, compare with the quantum affine algebra foldings in [Len14b].

In particular one would like to construct via some generalized folding the remaining exceptional cases of low-rank 2, 3 classified in [HV14] and known (and possibly new) cases of rank 1.

*Example 4.13.* In rank 3, apart from  $A_3, C_3$ , there is only one finite dimensional exceptional Nichols algebra over a nonabelian group (defined for any field with a third root of unity or any field characteristic 3). It is *not* of Lie type, but has the root system rank 3 no. 9 with 13 roots and root space dimensions (in the first object  $a$  with  $B_3$  Cartan matrix, see [HV14, Lm. 8.8]):

---

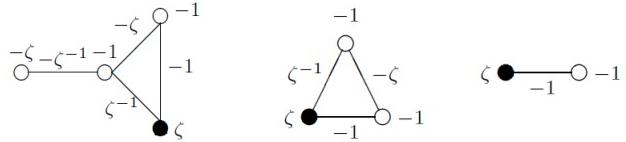
<sup>6</sup>For the readers convenience: The super Lie algebras (numbered 4-10 in Heckenberger's list) have root systems  $A_3, B_3, \mathcal{A}_3^1(2), \mathcal{A}_3^2(2)$  with respectively 6, 9, 7, 8 roots, as well as the sporadic root system labeled 6 with 13 roots. The rather alien Nichols algebras with numbers 16, 17 in Heckenberger's list have the sporadic root systems labeled 1, 2 with 10 resp. 11 roots.



$\alpha \in R_+^a$	$\dim(M_\alpha)$
$1, 2, 12, 12^2 3^4, 12^3 3^4, 1^2 2^3 3^4$	1
$23^2, 123^2, 12^2 3^2$	2
$3, 23, 123, 12^2 3^3$	3

The rank 2 parabolic generated by  $\alpha_2, \alpha_3$  is an exceptional Nichols algebra over a nonabelian group and has a standard root system of type  $B_2$ . The three-dimensional rank 1 parabolics are the Nichols algebras of dimension 12 over the nonabelian group  $\mathbb{S}_3$ , which was the first known Nichols algebra over a nonabelian group [MS00].

Ideally one would want to construct also these Nichols algebras via restriction, and we start our search with purely parabolic restrictions (no diagram automorphism). *Indeed* we find, that a restriction of the sporadic root system of rank 4 no. 7 with 25 roots (and the parabolics rank 3 no. 2 and  $B_2$ ) have precisely the right multiplicities. Moreover, they can be realized (in characteristic 0 with  $\zeta$  a primitive third root of unity) by an inclusion chain of diagonal Nichols algebras, namely row 17, row 16 and row 6 for  $q = -1$ :<sup>7</sup>



Hence we *can* obtain Nichols algebras with the right root system and root space dimensions in the category of  $\mathbb{C}[x]/(x^3)$ -Yetter-Drinfel'd modules. However the Hilbert series' do *not* coincide completely, e.g. in the smallest example we obtain a Nichols algebra with Hilbert series  $(2)_t(3)_t(6)_t$  instead of  $(2)_t(3)_t(2)_t$  over the group  $\mathbb{S}_3$ , which suggests to somehow consider a subalgebra of graded index  $(3)_{t^2}$ . If and how our Nichols algebras above can indeed be turned into Nichols algebras over the respective nonabelian groups is not clear at this point (for the folding in [Len14a] this was a central extension, but this cannot be true for  $\mathbb{S}_3$ ).

---

<sup>7</sup>Images from Heckenberger's list modified by adding a black dot.

**4.3. Restrictions of Nichols algebras.** Let  $\mathcal{B}(M^a)$  be the Nichols algebra of a semisimple object  $M^a = M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_n}$  in a braided category  $\mathcal{C}$ , and assume for now  $\mathcal{C} = {}^h\mathcal{YD}$ . Let  $(\mathcal{A}, R)$  be the associated  $n$ -dimensional crystallographic arrangement where the chamber  $a$  has positive simple roots  $\Delta^a = \{\alpha_1, \dots, \alpha_n\}$ .

Fix some  $\alpha_i$ . Recall that in the construction of the reflection  $r_{\alpha_i}$  in Section 4.2 we have considered the algebra of coinvariants  $\mathcal{B}(\bar{M})$  of the Hopf algebra map  $\pi : \mathcal{B}(M^a) \rightarrow \mathcal{B}(M_{\alpha_i})$ , which is a Hopf algebra in a different braided category

$$\mathcal{B}(\bar{M}) := \mathcal{B}(M^a)^{\text{coin } \mathcal{B}(M_i)} \in \frac{\mathcal{B}(M_{\alpha_i})}{\mathcal{B}(M^a)} \mathcal{YD}(\mathcal{C}),$$

and by Theorem 4.5 it is a Nichols algebra  $\mathcal{B}(\bar{M})$  in this category and thus has an associated root system. The aim of the next theorem is to show that the root system of  $\mathcal{B}(\bar{M})$  is precisely the restricted root system  $(\mathcal{A}^X, \bar{R})$  with  $X = \alpha_i^\perp$ . It moreover gives interpretations for the restriction multiplicities and possible non-reducedness of  $(\mathcal{A}^X, \bar{R})$  in terms of  $\mathcal{B}(\bar{M})$ . Repeating this argument for a subset  $J$  of simple roots, we get a similar statement for an arbitrary intersection of hyperplanes  $X = J^\perp$ .

Let us give in advance a *heuristic argument*, that underlies the proof below: The restriction to  $X$  induces a map of sets (additive where appropriate) between the set of roots  $R \rightarrow \bar{R} \cup \{0\}$  and we denote by  $\bar{\alpha} \in \bar{R}_+^a \cup \{0\}$  the image of a root  $\alpha \in R_+^a$ .

- The preimage of 0 are all roots  $\alpha$  where  $X \subset \alpha^\perp$ , i.e. in terms of arrangements the localization  $\mathcal{A}_X$ , i.e. in terms of roots the parabolic subsystem generated by  $J$ .

In the PBW-basis of  $\mathcal{B}(M) = \bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha)$  the respective factors  $\mathcal{B}(M_\beta)$  for all  $\beta$  with  $\bar{\beta} = 0$  hence form precisely the PBW-basis of the subalgebra  $\mathcal{B}(M_J)$  with  $M_J = \bigoplus_{i \in J} M_{\alpha_i}$ .

- The preimage of any  $\bar{\alpha} \neq 0$  is a root  $\alpha$  up to addition of roots in  $J$ .

Hence *morally*, adding all preimages of  $\bar{\alpha}$  should yield an irreducible  $\mathcal{B}(M_J)$ -Yetter-Drinfel'd submodule

$$\bar{M}_{\bar{\alpha}} := \bigoplus_{\beta, \bar{\beta}=\bar{\alpha}} M_\beta$$

of  $\bar{M}$ . The rank of  $\bar{M}$ , as  $h$ -Yetter-Drinfel'd module, is by construction equal to the number of preimages of  $\bar{\alpha}$  and hence the restriction multiplicity.

However, since in general  $M_{\alpha_i+\beta} \neq \text{ad}_{M_{\alpha_i}}(M_\beta)$  this is *not* true, but we can argue similarly with other  $\bar{M}_{\bar{\alpha}}$  defined via the right hand side expression.

- Finally, in the PBW-decomposition of  $\mathcal{B}(\bar{M})$  only the roots  $\bar{\alpha}$  with  $\frac{1}{k}\bar{\alpha} \notin \bar{R}_+$  appear. The other factors  $\mathcal{B}(M_\beta), \bar{\beta} \in \mathbb{N}\bar{\alpha}$  appear implicitly as higher commutators of  $\mathcal{B}(\bar{M}_{\bar{\alpha}})$ . This is why we work with the non-reduced system.

Finally this yields a PBW-decomposition of  $\mathcal{B}(\bar{M})$  according to  $(\mathcal{A}^X, \bar{R}_+^a)$ :

$$\mathcal{B}(\bar{M}_{\bar{\alpha}}) = \bigotimes_{\beta, \bar{\beta} \in \mathbb{N}\bar{\alpha}} \mathcal{B}(M_\beta), \quad \mathcal{B}(\bar{M}) = \bigotimes_{\bar{\alpha} \in \bar{R}_+^a} \mathcal{B}(\bar{M}_{\bar{\alpha}}).$$

We now turn to the exact proof:

#### Theorem 4.14.

- a) Assume  $M^a = M_i \oplus M_j$  is of rank 2 and  $\mathcal{B}(M)$  is finite-dimensional. Then there is an isomorphism of  $\mathbb{N}$ -graded  $\frac{\mathcal{B}(M_i)}{\mathcal{B}(M_j)}\mathcal{YD}(\mathcal{C})$ -objects

$$\bigotimes_{\alpha_i \neq \alpha \in R_+^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(\bar{M}_{\bar{\alpha}_j}), \quad \bar{M}_{\bar{\alpha}_j} := \text{ad}_{\mathcal{B}(M_i)}(M_j).$$

(Note that the tensor factors are not  $\mathcal{B}(M_i)$ -Yetter-Drinfel'd modules.)

- b) Under the assumptions in a) the restricted root systems of  $\mathcal{A}^{\alpha_i^\perp}$  and of  $\mathcal{B}(\bar{M})$  are trivially coinciding (type  $A_1$ ), but moreover the restriction multiplicity of  $\bar{\alpha}_j \in \bar{R}_+^a$  equals the rank of the new simple summand  $\bar{M}_{\bar{\alpha}_j}$  as semisimple object in  $\mathcal{C}$ . Non-reducedness, i.e.  $2\bar{\alpha}_j \in \bar{R}_+^a$ , implies  $[\bar{M}_{\bar{\alpha}_j}, \bar{M}_{\bar{\alpha}_j}] \neq 0$ .
- c) Let  $M^a = \bigoplus_{i \in I} M_i$  and assume  $\mathcal{B}(M)$  is finite-dimensional with root system  $R_+^a$ . For  $J \subset I$  let  $M_J = \bigoplus_{i \in J} M_i$  and let  $(\mathcal{A}^X, \bar{R}), X = J^\perp$  be the restricted root system. Then there exists an isomorphism

$$\bigotimes_{\bar{\alpha} \in \bar{R}_+^a} \mathcal{B}(\bar{M}_{\bar{\alpha}}) \cong \mathcal{B}(M^a)^{\text{coin } \mathcal{B}(M_J)}$$

of  $\mathbb{N}^{I \setminus J}$ -graded objects in  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M)}\mathcal{YD}(\mathcal{C})$  for suitable  $\bar{M}_{\bar{\alpha}} \in \frac{\mathcal{B}(M_J)}{\mathcal{B}(M)}\mathcal{YD}(\mathcal{C})$  where in particular  $\bar{M}_{\bar{\alpha}_j} := \text{ad}_{\mathcal{B}(M_J)}(M_j)$  for  $j \notin J$ . Moreover, the restriction multiplicity of  $\bar{\alpha}$  determines the rank of the new simple summand  $M_{\bar{\alpha}}$  as semisimple object in  $\mathcal{C}$  and non-reducedness, i.e.  $2\bar{\alpha} \in \bar{R}_+^a$  implies  $[\bar{M}_{\bar{\alpha}}, \bar{M}_{\bar{\alpha}}] \neq 0$ .

- d) Under the assumptions in c) for the Nichols algebra  $\mathcal{B}(\bar{M}) := \mathcal{B}(M^a)^{\text{coin } \mathcal{B}(M_J)}$  in the category  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M)}\mathcal{YD}(\mathcal{C})$ , the associated root system (crystallographic arrangement) in the sense of the Theorem 4.5 coincides with the reduced crystallographic arrangement of the restricted root system  $(\mathcal{A}^X, \bar{R})$  with  $X = J^\perp$  in the sense of Lemma 3.3.

*Proof.*

a) By Theorem 4.5 (PBW basis) we have an isomorphism of  $\mathbb{N}^2$ -graded  $\mathcal{C}$ -objects

$$\bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(M),$$

and since the map is multiplication in  $\mathcal{B}(M)$  it is an isomorphism in  $\frac{\mathcal{B}(M_i)}{\mathcal{B}(M_i)} \mathcal{YD}(\mathbb{C})$  (of course the tensor factors are no submodules in this matter).

By the same theorem we have an isomorphism of  $\mathbb{N}$ -graded  $\frac{\mathcal{B}(M_i)}{\mathcal{B}(M_i)} \mathcal{YD}(\mathcal{C})$ -objects

$$\mathcal{B}(\text{ad}_{\mathcal{B}(M_{\alpha_i})}(M_{\alpha_j})) \cong \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_i)}.$$

By the Radford projection theorem we have an isomorphism of  $\mathcal{C}$ -algebras

$$\mathcal{B}(M) \cong \mathcal{B}(M_i) \otimes \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_i)},$$

$$x \mapsto \pi(x^{(1)}) \otimes \pi(S(x^{(2)}))x^{(3)},$$

$$ab \leftrightarrow a \otimes b.$$

and since the  $\mathbb{N}$ -grading in the rank 1 Nichols algebra  $\mathcal{B}(\text{ad}_{\mathcal{B}(M_{\alpha_i})}(M_{\alpha_j}))$  is given by the radical filtration and the  $\mathcal{B}(M_i)$ -action by the adjoint action, this is even an isomorphism of  $\mathbb{N}$ -graded  $\frac{\mathcal{B}(M_i)}{\mathcal{B}(M_i)} \mathcal{YD}(\mathcal{C})$ -objects.

Altogether we get a surjective map of  $\mathbb{N}$ -graded  $\frac{\mathcal{B}(M_i)}{\mathcal{B}(M_i)} \mathcal{YD}(\mathcal{C})$ -objects

$$\bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha) \xrightarrow{\text{mult.}} \mathcal{B}(M) \xrightarrow{x \mapsto \pi(S(x^{(1)}))x^{(2)}} \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_i)} \cong \mathcal{B}(\text{ad}_{\mathcal{B}(M_i)}(M_j)).$$

We immediately see that  $\cdots \otimes \mathcal{B}(M_i)^+$  is in the kernel. Hence this factorizes to the following surjective map, which is even an isomorphism since the dimensions are equal:

$$\bigotimes_{\alpha_i \neq \alpha \in R_+^a} \mathcal{B}(M_\alpha) \xrightarrow{\cong} \mathcal{B}(\text{ad}_{\mathcal{B}(M_i)}(M_j)).$$

b) By definition

$$\bar{M} = \text{ad}_{\mathcal{B}(M_i)}(M_j) = \bigoplus_{k=0}^{\infty} \text{ad}_{M_i}^k(M_j) = \bigoplus_{k=0}^{-c_{ij}} \text{ad}_{M_i}^k(M_j)$$

and by Theorem 4.5 all  $\text{ad}_{M_i}^k(M_j)$  are simple  $\mathcal{C}$ -objects. Hence the rank of  $\bar{M}$  as  $\mathcal{C}$ -object is  $1 - c_{ij}$ . On the other hand the roots restricting to  $\bar{\alpha}_j$  in  $\mathcal{A}^{\alpha_i^\perp}$  are all elements in the root string  $\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j - c_{ij}\alpha_i$ , so the two numbers coincide as claimed. The claim about non-reducedness comes from considering the dimensions, since  $\bar{M}$  generates  $\mathcal{B}(\bar{M})$ .

c) Since both parabolic restriction and taking coinvariants can be performed root-by-root it suffices to treat the case  $J = \{\alpha_1\}$ . As in a) we have by the PBW basis theorem an isomorphism of  $\mathbb{N}^{I \setminus J}$ -graded objects in  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M)} \mathcal{YD}(\mathcal{C})$

$$\bigotimes_{\alpha \in R_+^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(M^a)$$

and by the Radford projection theorem an isomorphism of  $\mathbb{N}^{I \setminus J}$ -graded objects in  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M)} \mathcal{YD}(\mathcal{C})$

$$\bigotimes_{\alpha \in R_+^a \setminus R_{J,+}^a} \mathcal{B}(M_\alpha) \cong \mathcal{B}(M^a)^{\text{coin } \mathcal{B}(M_J)}$$

where  $R_{J,+}^a := R_+^a \cap J\mathbb{N}$  is the parabolic subsystem (localization). Now restriction to  $(\mathcal{A}^{J^\perp}, \bar{R})$  defines a surjective map (even compatible with addition where appropriate)

$$R_+^a \setminus R_{J,+}^a \rightarrow \bar{R}_+^a$$

where the preimages of  $\bar{\alpha}_j$  is the root string  $(\alpha_j + \mathbb{N}\alpha_i) \cap R_+^a$ . Similarly all other preimages of  $\bar{\alpha}$  are root strings after reflection, i.e. in some other object  $a'$  where  $\alpha$  is in a parabolic subsystem  $\langle \alpha_i, \alpha_j \rangle$ . Reordering (which is clearly isomorphism) and using for each  $\bar{\alpha}$  the rank 2 result of a) yields the desired isomorphism. The claims on multiplicities and nonreducedness follow from c).

d) Since we obtained in c) a decomposition

$$\bigotimes_{\bar{\alpha} \in \bar{R}_+^a} \mathcal{B}(M_{\bar{\alpha}}) \cong \mathcal{B}(\bar{M})$$

of the Nichols algebra in  $\mathbb{N}^{I \setminus J}$ -graded  $\mathcal{B}(M_J)$ -Yetter-Drinfel'd modules, and the degrees of this decomposition is by construction given by  $\mathcal{A}^X$ , we can apply the uniqueness result in [HS10] Theorem 4.5 (2).<sup>8</sup>  $\square$

**4.4. First examples of restrictions.** We first discuss very explicitly two small examples of parabolic restriction of diagonal Nichols algebras, which yield familiar restricted root systems.

For the convenience of the reader we first compare the notation for the diagonal case in Lusztig's book on quantum groups [Lusz94] and Heckenberger's paper [Heck07] on reflections in diagonal Nichols algebras:

---

<sup>8</sup>We thank Jing Wang for pointing out this uniqueness result for the root system.

Diagonal case	General case
Braided vector space spanned by $\theta_i$ , $E_i$ or $x_i$ for $i \in I$ $V = V^+(\chi)$ depending on a bicharacter $\chi$ .	Yetter-Drinfel'd module $M$
Diagonal braiding $x_i \otimes x_j \rightarrow q_{ij} x_j \otimes x_i$ braiding matrix $q_{ij} = v^{i \cdot j / 2} = q^{d_i c_{ij}}$ or more generally $q_{ij} = \chi(\alpha_i, \alpha_j)$	Yetter-Drinfel'd module braiding $a \otimes b \mapsto a^{(-1)}.b \otimes a^{(0)}$
1-dim. subspaces $\theta_i \mathbb{C}$ , $E_i \mathbb{C}$ , $x_i \mathbb{C}$ Duals $F_i \mathbb{C}$ in $V^* = V^-(\chi)$	Simple summands $M_i$ of $M$ Duals $M_i^*$ in $M^*$
Nichols algebra $\mathbf{f}$ , $\mathcal{B}(V)$ or $\mathcal{U}^+(\chi)/\mathcal{I}^+$	Nichols algebra $\mathcal{B}(M)$
Cartan matrix $-c_{ij} = \max\{m \mid \text{ad}_{x_i}^m(x_j) \neq 0\}$ $= \min\{m \mid (m+1)_{q_{ii}} = 0 \text{ or } q_{ii}^m q_{ij} q_{ji} = 1\}$	Cartan matrix $-c_{ij} := \max\{m \mid \text{ad}_{M_i}^m(M_j) \neq 0\}$
Subalgebras $\mathbf{f}[i]$ in [Lusz94] Sec. 38.1.1 generated by all $\text{ad}_{\theta_i}^m(\theta_j)$ for fixed $i \in I$ Subalgebra $\mathcal{U}_{+i}^+(\chi)$ modulo $\mathcal{I}^+$ for $i \in I$ in [Heck07] Def. 4.29, Prop. 5.10	Coinvariants over $\mathcal{B}(M_i)$ for $i \in I$ $K_i := \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_i)}$
Set of positive roots $R^+$ , $\mathcal{R}_+^a$ Reflection operator $r_i$ , $T_i$ for $i \in I$	Set of positive roots $\mathcal{R}_+^a$ Reflection operator $T_i$ for $i \in I$

*Example 4.15.* Let  $M_1 = x_1 \mathbb{C}$  and  $M_2 = x_2 \mathbb{C}$  be Yetter-Drinfel'd modules over  $\Gamma := \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g_1, g_2 \rangle$  with

$$\chi_1(g_1) = -1, \quad \chi_2(g_1) = -1,$$

$$\chi_1(g_2) = 1, \quad \chi_2(g_2) = -1.$$

Then  $M := M_1 \oplus M_2$  has diagonal braiding

$$q_{ij} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

and the Nichols algebra  $\mathcal{B}(M)$  is of Lie type  $A_2$ ,

$$R_+^a = \{(1, 0), (0, 1), (1, 1)\}.$$

This means that multiplication in the Nichols algebra induces the following isomorphism of graded vector spaces (PBW-basis)

$$\mathcal{B}(x_1) \otimes \mathcal{B}(x_{12}) \otimes \mathcal{B}(x_2) \cong \mathcal{B}(M) \quad x_{12} := x_1 x_2 + x_2 x_1 \in \mathcal{B}(M).$$

The Hilbert series is hence  $(2)_t^2(2)_{t^2}$  and the dimension is  $2^3$ .

Consider the coinvariants  $K_1 = \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_1)}$  with respect to the projection  $\mathcal{B}(M_1 \oplus M_2) \rightarrow \mathcal{B}(M_1)$ , it is generated by  $x_2, x_{12}$  (this is an easy direct calculation). By Theorem 4.5,  $K_1$  is the Nichols algebra of the  $\mathcal{B}(M_1)$ -Yetter-Drinfel'd module<sup>9</sup>

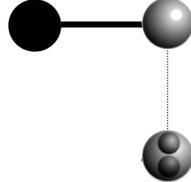
$$\bar{M} = \text{ad}_{\mathcal{B}(M)}(N) = \text{ad}_1(x_2) + \text{ad}_{x_1}(x_2) = x_2 \mathbb{C} \oplus x_{12} \mathbb{C}.$$

$\bar{M}$  is simple as a  $\mathcal{B}(M_1)$ -Yetter-Drinfel'd module (hence the root system of  $\mathcal{B}(\bar{M})$  is of type  $A_1$ ) and  $\bar{M}$  is semisimple of rank 2 as an object in the base category  ${}_\Gamma^{\Gamma}\mathcal{YD}$  (since  $\Gamma$  is abelian, this is equal to the  $\mathbb{C}$ -dimension).

For the convenience of the reader we write down the explicit *nondiagonal braiding* on  $\bar{M}$  as it follows from  $\mathcal{B}(M_1)$ -action and -coaction; observe that the off-diagonal term comes from a  $2x_1$  coacting from  $x_{12}$  and then acting on  $x_2$ :

$$\begin{pmatrix} x_2 \otimes x_2 & x_2 \otimes x_{12} \\ x_{12} \otimes x_2 & x_{12} \otimes x_{12} \end{pmatrix} \mapsto \begin{pmatrix} & -x_2 \otimes x_2 & -x_{12} \otimes x_2 \\ & -2x_{12} \otimes x_2 + x_2 \otimes x_{12} & -x_{12} \otimes x_{12} \end{pmatrix}$$

This is in agreement with our result. Namely, the restriction of the root system  $(\mathcal{A}, R)$  of type  $A_2$  to the hyperplane  $H_{\alpha_1}$  is the root system  $(\mathcal{A}^{H_{\alpha_1}}, \bar{R})$  of type  $A_1$ . More precisely,  $\alpha_1 = (1, 0)$  restricts to  $0 \in \bar{R}$  and is removed, while both  $\alpha_2, \alpha_{12}$  restrict to the same simple root in  $\bar{R}$  (multiplicity 2).



As indicated in the application part of the introduction, this  $\bar{M}$  is the fundamental  $\mathfrak{sl}_2$ -representation used in [Rosso98, Sec. 4] as an induction step from  $u_q(\mathfrak{sl}_2)$  to  $u_q(\mathfrak{sl}_3)$ . In the same way our result explains the different  $A_{n-1}$  representations appearing in Rosso's induction step from  $A_{n-1}$  to  $A_n, B_n, C_n, D_n$ , etc.

We next consider an example which is not of Lie type:

---

<sup>9</sup>inside the base category  ${}_\Gamma^{\Gamma}\mathcal{YD}$  i.e.  $K_1$  is actually a  $\mathcal{B}(M_1) \rtimes k[\Gamma]$ -Yetter-Drinfel'd module.

*Example 4.16.* Consider the diagonal Nichols algebra  $\mathcal{B}(M)$  of rank 3, thoroughly treated in Example 4.8, which has braiding matrix

$$q_{\alpha_i \alpha_i} = -1, \quad q_{\alpha_i \alpha_j} q_{\alpha_j \alpha_i} = \zeta,$$

and the following root system and vector space PBW-basis:

$$\begin{aligned} R_+^a &= \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}, \\ \mathcal{B}(M) &\cong \mathcal{B}(x_{\alpha_1}) \otimes \mathcal{B}(x_{\alpha_2}) \otimes \mathcal{B}(x_{\alpha_3}) \otimes \mathcal{B}(x_{\alpha_{12}}) \otimes \mathcal{B}(x_{\alpha_{13}}) \otimes \mathcal{B}(x_{\alpha_{23}}) \otimes \mathcal{B}(x_{\alpha_{123}}) \\ &\cong \mathbb{C}[x_{\alpha_1}]/(x_{\alpha_1}^2) \otimes \mathbb{C}[x_{\alpha_2}]/(x_{\alpha_2}^2) \otimes \mathbb{C}[x_{\alpha_3}]/(x_{\alpha_3}^2) \\ &\quad \otimes \mathbb{C}[x_{\alpha_{12}}]/(x_{\alpha_{12}}^3) \otimes \mathbb{C}[x_{\alpha_{13}}]/(x_{\alpha_{13}}^3) \otimes \mathbb{C}[x_{\alpha_{23}}]/(x_{\alpha_{23}}^3) \\ &\quad \otimes \mathbb{C}[x_{\alpha_{123}}]/(x_{\alpha_{123}}^2). \end{aligned}$$

Consider the projection to the Nichols algebra

$$\mathcal{B}(M_{\alpha_2}) \cong \mathbb{C}[x_{\alpha_2}]/(x_{\alpha_2}^2).$$

The action and coaction of  $x_{\alpha_2}$  span the following simple 2-dimensional Yetter-Drinfel'd submodules over  $\mathcal{B}(M_{\alpha_2}) \rtimes \mathbb{C}[\Gamma]$ :

$$\begin{aligned} M_{\bar{\alpha}_1} &:= M_{\alpha_1} \oplus M_{\alpha_{12}}, \\ M_{\bar{\alpha}_2} &:= M_{\alpha_3} \oplus M_{\alpha_{23}}, \\ M_{\bar{\alpha}_{12}} &:= M_{\alpha_{13}} \oplus M_{\alpha_{123}}. \end{aligned}$$

We obtain the new generating Yetter Drinfel'd module according to Theorem 4.5:

$$\mathcal{B}(M)^{\text{coin } \mathcal{B}(M_{\alpha_2})} = \mathcal{B}(\bar{M}), \quad \bar{M} = \text{ad}_{\mathcal{B}(M_{\alpha_2})}(M_{\alpha_1} \oplus M_{\alpha_3}) = M_{\bar{\alpha}_1} \oplus M_{\bar{\alpha}_2}.$$

Altogether, this agrees with the crystallographic arrangement  $(\mathcal{A}^{H_{\alpha_2}}, \bar{R})$  which is of type  $A_2$  and all multiplicities 2. We get a new PBW-basis and Hilbert series

$$\begin{aligned} \mathcal{B}(\bar{M}) &\cong \mathcal{B}(M_{\bar{\alpha}_1}) \otimes \mathcal{B}(M_{\bar{\alpha}_2}) \otimes \mathcal{B}(M_{\bar{\alpha}_{12}}), \\ \mathcal{H}_{\mathcal{B}(\bar{M})}(t) &= (2)_t^2 (3)_t^2 (3)_{t^2} (2)_{t^2}. \end{aligned}$$

**4.5. Nichols algebras with new Weyl groupoids.** Recall from Corollary 4.11 that only certain root systems appear as root systems of Nichols algebras in  $\mathbb{C}^{[G]} \mathcal{YD}$ . Parabolic restriction produces Nichols algebras in other braided categories

$$\mathcal{B}(\bar{M}) := \mathcal{B}(M)^{\text{coin } \mathcal{B}(M_J)} \in {}_{\mathcal{B}(M_J)}^{\mathcal{B}(M_J)} \mathcal{YD}(\mathcal{C})$$

and by Theorem 4.14 the root system and Weyl groupoid corresponds to the restricted root system  $(\mathcal{A}^X, \bar{R})$ ,  $X = J^\perp$ . We hence find an answer to Question 1.1:

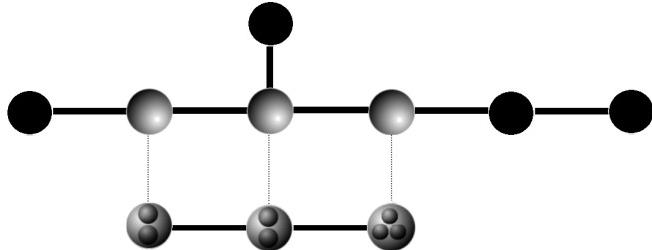
**Theorem 4.17.**

- (1) *There exist Nichols algebras whose corresponding crystallographic arrangement are the sporadic arrangements of rank three labeled 7, 13, 14, 15, 20, 23, although such a Nichols algebra does not exist over any finite group. These have 13, 16, 17, 17, 19, 19 positive roots respectively, and are restrictions of the reflection arrangements of type  $E_7, E_8, E_8, E_8, E_8, E_8$  respectively.*
- (2) *Since every crystallographic arrangement of rank greater than three is a (parabolic) restriction of a Weyl arrangement, every crystallographic arrangement of rank greater than three is symmetry structure of some Nichols algebra.*

*Question 4.18.* Do all root systems appear as root systems of some Nichols algebra in some braided category  $\mathcal{C}$ ?

We now give first an example with a new sporadic Weyl groupoid of rank three labeled 7:

*Example 4.19.* Let  $\mathcal{B}(M) = u_q(E_7)^+$ , which is a diagonal Nichols algebra of dimension  $\text{ord}(q^2)^{63}$ . We consider the parabolic restriction indicated in the diagram:



Then the restriction of the Weyl group arrangement  $E_7$  with 63 roots has a Weyl groupoid of sporadic type 7 and 13 roots (one of which is nonreduced), namely

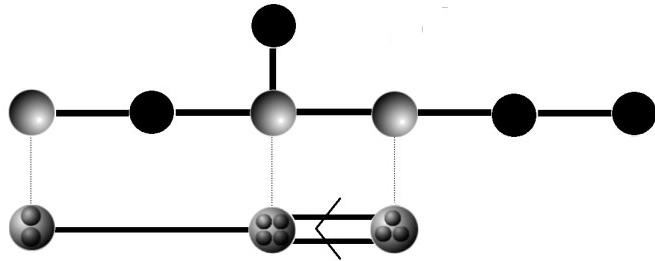
$R_+^a$	$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$	$(1, 1, 0)$	$(0, 1, 1)$	$(1, 1, 1)$	$+2(1, 1, 1)$
multiplicity	2	2	3	4	6	12	+3
$R_+^a$	$(1, 2, 1)$	$(2, 2, 1)$	$(1, 2, 2)$	$(2, 3, 2)$	$(2, 3, 3)$	$(2, 4, 3)$	$(3, 4, 3)$
multiplicity	6	3	6	6	2	1	2

This yields a Nichols algebra  $\mathcal{B}(\bar{M})$  of dimension  $\text{ord}(q^2)^{58}$  over an object  $\bar{M}$  of rank 3 and dimension  $2 + 2 + 3$  in the braided category  $\frac{\mathcal{B}(M_J)}{\mathcal{B}(M_J)}\mathcal{YD}$  with  $J = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7\}$ , where  $\mathcal{B}(M_J)$  is an ordinary Borel part  $u_q(A_1 \times A_1 \times A_2)^+$ .

This root system is clearly not one of Lie type. This means that reflections in the Weyl groupoid connect the Weyl chamber described above with other chambers corresponding possibly to different Dynkin diagrams and non-isomorphic Nichols algebras (compare Example 4.8).

For the restriction these different choices arise from different parabolics  $J'$  of  $E_7$  in the Weyl group orbit of the choice  $J$  above. To be more precise in our example: The sporadic Weyl groupoid of rank three labeled 7 has by [CH15] Table 1 altogether 12 different<sup>10</sup> chambers (objects) with the following Dynkin diagrams (see the Figure below for the object change diagram):  $A_3, B_3, C_3, \mathcal{A}_3^1(2)$  and the sporadic Dynkin diagrams  $\Gamma_3^1, \Gamma_3^7, \Gamma_3^{11}, \Gamma_3^{15}$  in [CH15].

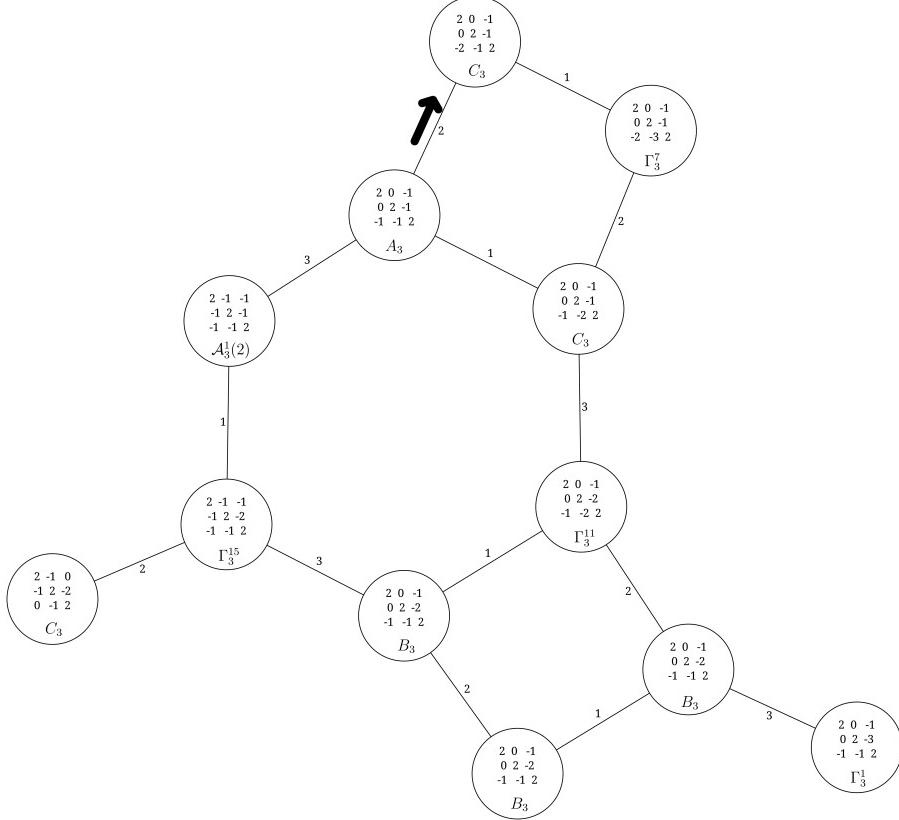
How do these different objects in the restriction arise, even though  $E_7$  itself has only one type of chamber? This is because the parabolic  $J$  is not fixed by the Weyl group of  $E_7$ . For example, the reflections in  $E_7$  on the leftmost nodes  $s_{\alpha_3}s_{\alpha_1}s_{\alpha_3}$  map  $J$  above to the following  $J'$ . In the parabolic restriction this changes the object (chamber) above with Dynkin diagram  $A_3$  to a different object (chamber) with Dynkin diagram  $C_3$ . In particular the now simple root with multiplicity 4 is the root  $(1, 1, 0)$  above and the double edge arises from the root string  $(0, 0, 1), (1, 1, 1), (2, 2, 1)$ .




---

<sup>10</sup>To be precise, this table only shows 12 different root sets. Counting different multiplicities of the roots, the number of objects could a-priori multiply up to 96 (simply connected Weyl groupoid). But in this example, there are 12 parabolics  $J' \cong A_1 \times A_1 \times A_2$  restricting to no. 7.

We finally present the full *object change diagram* of the sporadic Weyl groupoid of rank three labeled 7. It shows Cartan matrix and Dynkin diagram in each object (chamber) and the edges are labeled by reflections on the  $i$ -th simple root (simple roots in the standard ordering, differently from the ordering above!). Reflections not drawn map to the same object. The thick arrow shows the reflection of the  $J$ -restriction to the  $J'$ -restriction discussed above.



## 5. A SIMPLICIAL COMPLEX OF NICHOLS ALGEBRAS

**5.1. The simplicial complex.** As in [CMW15], to a crystallographic arrangement  $(\mathcal{A}, R)$  of rank  $n$  we associate the *simplicial complex*  $S$  on  $V$  whose  $(n-1)$ -cells are the chambers of  $\mathcal{A}$  intersected with the  $n-1$ -sphere. This complex  $S \subseteq V$  is topologically isomorphic to a sphere. Sometimes it will be convenient to add a single  $(-1)$ -cell corresponding to the origin  $0 \in V$ . We need the following facts (see [CMW15] for proofs and more details):

**Lemma 5.1.**

- a) Let  $K$  be a chamber and  $\Delta^K$  be the associated set of simple roots. Then the adjacent  $(k-1)$  cells  $x \subseteq K$  are in bijection with the subsets  $J_{x,K} \subseteq \Delta^K$  of order  $n-k$ . Moreover  $x \subseteq X_x := J_{x,K}^\perp$  is of dimension  $k$ .
- b) Let  $K_1 \neq K_2$  be chambers and let  $w$  be the Weyl groupoid element with  $w.K_1 = K_2$ . Then the intersection  $K_1 \cap K_2$  is a common adjacent  $(k-1)$ -cell  $x \subseteq K_1, K_2$ ,  $w$  is contained in the Weyl groupoid of the localization  $\mathcal{A}_{X_x}$ , and  $w.J_{x,K_1} = J_{x,K_2}$ .

**5.2. The decoration by Nichols algebras in braided categories.** Suppose we have a Nichols algebra  $\mathcal{B}(M)$ , for now in the braided category  $\mathcal{C} = {}^h\mathcal{YD}$ . We consider the crystallographic arrangement  $(\mathcal{A}, R)$  with roots  $\alpha \in R$  labeled by objects  $M_\alpha \in \mathcal{C}$  from Theorem 4.5. More precisely, the initial Nichols algebra is  $\mathcal{B}(M^a)$  associated to some chamber  $a$  of  $\mathcal{A}$  and one set of positive simple roots  $\Delta_a^+$  such that  $M^a = \bigoplus_{\alpha \in \Delta_a^+} M_\alpha$ , while the other chambers correspond to Weyl equivalent Nichols algebras  $\mathcal{B}(M^{a'})$ . Thus it would be better to think of  $(\mathcal{A}, R)$  being associated to the entire Weyl equivalence class and a specific representing Nichols algebra  $\mathcal{B}(M^a) \in \mathcal{C}$  being associated to each  $(n-1)$ -cell  $a$ . We now wish to associate Nichols algebra data to all  $(k-1)$ -cells  $x$ .

*Definition 5.2.* As in Lemma 5.1, let  $x$  be a  $(k-1)$ -cell, let  $X_x \in L(\mathcal{A})$  the associated subspace of  $V$ , let  $a$  be any adjacent chamber with simple roots  $\Delta^a$  and  $J_{x,a} \subseteq \Delta^a$ .

Then we associate to the pair  $(x, a)$  the braided category of the localized Nichols algebra

$$\mathcal{C}_x := \frac{\mathcal{B}(M_{x,a})}{\mathcal{B}(M_{x,a})}\mathcal{YD}, \quad M_x := \bigoplus_{\alpha \in J_{x,a}} M_\alpha$$

and the restricted Nichols algebra in this category as in Theorem 4.14

$$\mathcal{B}(M)^x := \mathcal{B}(M^a)^{\text{coin } \mathcal{B}(M_{x,a})} \cong \mathcal{B}(M^{x,a}), \quad M^{x,a} := \text{ad}_{\mathcal{B}(M_{x,a})} \left( \bigoplus_{\alpha \in \Delta^a \setminus J_{x,a}} M_\alpha \right).$$

*Remark 5.3.* The previous definition is *independent* of the chosen adjacent chamber  $a$  in the following sense: Let  $a_1, a_2$  be chambers adjacent to  $x$ , then by Lemma 5.1 b) we have an element  $w$  in the Weyl groupoid of  $\mathcal{A}$ , which is contained in the localization to  $X_x$ , such that  $w.a_1 = a_2$  and  $w.J_{x,a_1} = J_{x,a_2}$ . Thus  $R_w(M_{x,a_1}) = M_{x,a_2}$  and the Nichols algebras  $\mathcal{B}(M_{x,a_1}), \mathcal{B}(M_{x,a_2})$  are Weyl equivalent. By [BLS14] Thm. 4.4 this implies there is a category isomorphism

$$r_w : \frac{\mathcal{B}(M_{x,a_1})}{\mathcal{B}(M_{x,a_1})}\mathcal{YD} \cong \frac{\mathcal{B}(M_{x,a_2})}{\mathcal{B}(M_{x,a_2})}\mathcal{YD},$$

$$r_w : \mathcal{B}(M^{x,a_1}) \mapsto \mathcal{B}(M^{x,a_2}).$$

In particular, the vector space, the root system and Dynkin diagram, and the vector space dimension of the nodes  $\bigoplus_i M_{\alpha_i} = M^{x,a}$  (multiplicities) are independent of  $a$  (in contrast to  $M_{x,a}$ ).

**Corollary 5.4.** *In this sense, the simplicial complex is decorated by sets of equivalent Nichols algebras in an equivalence class of braided categories and by Dynkin diagrams and root multiplicities.*

We first give the two extremal examples:

*Example 5.5.* The chambers  $x = a \in A$  are the  $(n - 1)$ -cells. Each corresponds to the empty subsets  $J_{a,a} = \{\}$  and is contained only in the chamber  $a$  itself. It is hence decorated with the unique category

$$\mathcal{C}_a = \frac{\mathcal{B}(M_{x,a})}{\mathcal{B}(M_{x,a})} \mathcal{YD} = {}^1_c \mathcal{YD} = 1_{\mathcal{C}}$$

and the following Nichols algebra in this category:

$$\mathcal{B}(M)^a = \mathcal{B}(M^a)^{\text{coin } 1_{\mathcal{B}(M^a)}} = \mathcal{B}(M^a) \in \mathcal{C}.$$

Hence as expected we associate to each  $(n - 1)$ -cell  $a$  the Nichols algebra  $\mathcal{B}(M^a)$  in the base category  $\mathcal{C}$ .

*Example 5.6.* The center point  $0$  is a  $(-1)$ -cell adjacent to any chamber  $a$ . It corresponds to the full set of simple roots depending on the chamber  $J_{0,a} = \Delta^a$  and is hence decorated with the equivalence class of categories

$$\mathcal{C}_0 = \frac{\mathcal{B}(M_{0,a})}{\mathcal{B}(M_{0,a})} \mathcal{YD} = \frac{\mathcal{B}(M^a)}{\mathcal{B}(M^a)} \mathcal{YD}$$

and the following Nichols algebras in each category:

$$\mathcal{B}(M)^0 = \mathcal{B}(M^a)^{\text{coin } \mathcal{B}(M^a)} = 1_{\frac{\mathcal{B}(M^a)}{\mathcal{B}(M^a)} \mathcal{YD}}.$$

Hence we associate to the  $(-1)$ -cell in the center the trivial Nichols algebras, i.e. on unit objects in all braided categories  $\frac{\mathcal{B}(M^a)}{\mathcal{B}(M^a)} \mathcal{YD}$  over some Weyl equivalence representative of the full Nichols algebra  $\mathcal{B}(M^a)$ .

We also note that *simple reflection* can be visualized nicely from this picture: Let  $a_1, a_2$  be adjacent chambers, meaning  $a_1 \cap a_2$  is an  $(n - 2)$ -cell, i.e. in a unique hyperplane  $H_\alpha, \alpha \in \Delta^{a_1}$ . Then the reflection  $r_\alpha : \mathcal{B}(M^{a_1}) \rightarrow \mathcal{B}(M^{a_2})$  constitutes of the following steps:

- Restriction to the hyperplane  $H_\alpha$  yields the restricted Nichols algebra  $K := \mathcal{B}(M^{a_1})^{\text{coin } \mathcal{B}(M_\alpha)} \in {}_{\mathcal{B}(M_\alpha)}^{\mathcal{B}(M_\alpha)} \mathcal{YD} = C_x$ .
- The orientation chance  $H_\alpha = H_{-\alpha}$  i.e.  $J_{x,a_1} = \{\alpha\} \mapsto \{-\alpha\} = J_{x,a_2}$  is simply a dualization  $\mathcal{B}(M_\alpha) \mapsto \mathcal{B}(M_\alpha^*)$ , yielding equivalent categories  $C_x = {}_{\mathcal{B}(M_\alpha)}^{\mathcal{B}(M_\alpha)} \mathcal{YD} \cong {}_{\mathcal{B}(M_\alpha^*)}^{\mathcal{B}(M_\alpha^*)} \mathcal{YD}$  and especially as image of  $K$  a Nichols algebra  $L \in {}_{\mathcal{B}(M_\alpha^*)}^{\mathcal{B}(M_\alpha^*)} \mathcal{YD}$ .
- Now  $K$  is the restriction of the reflected Nichols algebra  $\mathcal{B}(M^{a_2})$  to the hyperplane  $x = H_{-\alpha}$  and we may conversely obtain the reflected Nichols algebra from  $K$  by a Radford biproduct.

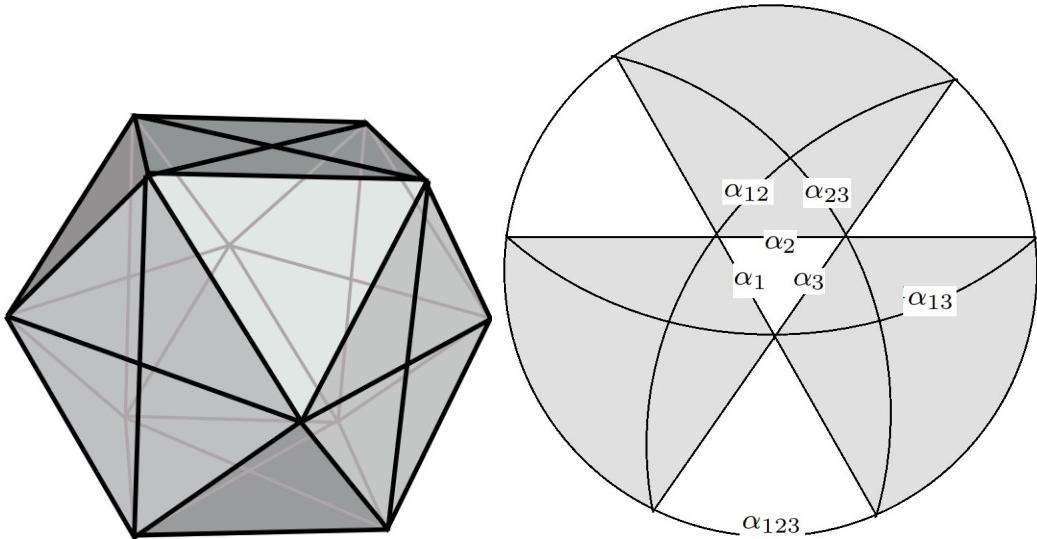
**5.3. Example: A Nichols algebra of rank 3.** Consider the diagonal Nichols algebra thoroughly treated in Example 4.8. It was of rank 3, and had two types of chambers  $a, a'$  with the following sets of 7 positive roots

$$R_+^a = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$$

$$R_+^{a'} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1), (1, 2, 1)\}$$

$$\text{where } \alpha'_1 = \alpha_{12}, \alpha'_2 = -\alpha_2, \alpha'_3 = \alpha_{23}$$

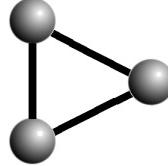
The following picture now shows the arrangement in perspective: It is basically a cuboctahedron (an archimedean solid) with 8 equilateral triangles and 6 squares, which has each square subdivided into 4 right triangles (by the three orthogonal planes  $H_{\alpha_{ij}}$ ). There are hence 7 hyperplanes and  $8 + 24 = 32$  chambers.



So we have a spherical simplicial complex of dimension  $3 - 1$ . We now calculate the categories and Nichols algebras associated to each  $(k - 1)$ -cell:

- 2-Cells: There are two types of chambers:

- There are 8 chambers associated to equilateral triangles as the example  $x = a$  bounded by  $H_{\alpha_1}, H_{\alpha_2}, H_{\alpha_3}$ . It is associated to the Nichols algebra  $\mathcal{B}(M^a)$  in the base category  $\mathcal{C}_x = \mathcal{C}$  with Dynkin diagram a cycle and Hilbert series  $(2)_t^3(3)_{t^2}^3(2)_{t^3}$



- There are 24 chambers associated to right triangles as in the example  $x = a'$  bounded by  $H_{\alpha_{12}}, H_{-\alpha_2}, H_{\alpha_{23}}$  (note  $H_{-\alpha_2} = H_{\alpha_2}$ ). It is associated to the Nichols algebra  $\mathcal{B}(M^{a'})$  in the base category  $\mathcal{C}_x = \mathcal{C}$  with Dynkin diagram  $A_2$  (but one more root) and Hilbert series  $(2)_t(3)_t^2(2)_{t^2}^2(2)_{t^3}(3)_{t^4}$ .



- 1-Cells: There are two types of edges:

- There are 24 edges between an equilateral and right triangle (the edges of the cuboctahedron) as in the examples  $a, a'$  the common hyperplane  $x = H_{\alpha_2}$ . It is associated to the following restricted Nichols algebra  $\mathcal{B}(M^{x,a})$  in the category

$$\mathcal{C}_x = \frac{\mathcal{B}(M_{\alpha_2})}{\mathcal{B}(M_{\alpha_2})} \mathcal{YD}, \text{ where } \mathcal{B}(M_{\alpha_2}) \cong \mathbb{C}[t_{\alpha_2}]/(t_{\alpha_2}^2)$$

Note that we calculate the restriction from the chamber  $a$ , but the calculation from  $a'$  would return the same result except  $\alpha_2 \rightarrow -\alpha_2$ . The generating Yetter-Drinfel'd modules is  $M^{x,a} = M_1^{x,a} \oplus M_2^{x,a}$ , the Nichols algebra is of Lie type  $A_3$ . More precisely, the simple summands are

$$M_1^{x,a} = M_{\alpha_1} \oplus M_{\alpha_{12}} \quad M_2^{x,a} = M_{\alpha_3} \oplus M_{\alpha_{23}},$$

$$[M_1^{x,a}, M_2^{x,a}] = M_{\alpha_{13}} \oplus M_{\alpha_{123}}.$$

In particular, the Hilbert series of  $\mathcal{B}(M^{x,a})$  is  $(2)_t^2(3)_t^2(3)_{t^2}(2)_{t^2}$ .



- There are 24 edges between two right triangles (half subdividing diagonals in the cuboctahedron squares) as in the examples  $a'$  the hyperplane  $x = H_{\alpha_{23}}$ . It is associated to the following restricted Nichols algebra  $\mathcal{B}(M^{x,a})$  in the category

$$\mathcal{C}_x = \frac{\mathcal{B}(M_{\alpha_{23}})}{\mathcal{B}(M_{\alpha_{23}})} \mathcal{YD}, \text{ where } \mathcal{B}(M_{\alpha_{23}}) \cong \mathbb{C}[t_{\alpha_{23}}]/(t_{\alpha_{23}}^3)$$

The generating Yetter-Drinfel'd modules is  $M^{x,a} = M_1^{x,a} \oplus M_2^{x,a}$ , the Nichols algebra is of type  $B_2$ . More precisely, the simple summands are

$$M_1^{x,a} = M_{-\alpha_2} \oplus M_{\alpha_1} \quad M_2^{x,a} = M_{\alpha_{23}}$$

$$[M_1^{x,a}, M_2^{x,a}] = M_{\alpha_3} \oplus M_{\alpha_{123}} \quad [M_1^{x,a}, [M_1^{x,a}, M_2^{x,a}]] = M_{\alpha_{13}}$$

In particular, the Hilbert series of  $\mathcal{B}(M^{x,a})$  is  $(2)_t^2(3)_t(2)_{t^2}^2(3)_{t^3}$ .



- 0-Cells: There are two types of 0-cells:

- There are 12 vertices with altogether three intersecting hyperplanes (the cuboctahedron vertices) as in the examples  $a$  the bounding hyperplanes  $x = H_{\alpha_1} \cap H_{\alpha_3}$ . It is associated to the following restricted Nichols algebra  $\mathcal{B}(M^{x,a})$  in the category

$$\mathcal{C}_x = \frac{\mathcal{B}(M_{\alpha_1} \oplus M_{\alpha_3})}{\mathcal{B}(M_{\alpha_1} \oplus M_{\alpha_3})} \mathcal{YD}$$

where  $\mathcal{B}(M_{\alpha_1} \oplus M_{\alpha_3})$  is a Nichols algebra of dimension  $2^2 3 = 12$  with Dynkin diagram  $A_2$  (the quantum Borel part of  $\mathfrak{sl}(2|1)$  at the third root of unity  $q = \zeta$ ): The generating Yetter-Drinfel'd module  $M^{x,a}$  is simple, the Nichols algebra of type  $A_1$ :

$$M^{x,a} = M_{\alpha_2} \oplus M_{\alpha_{12}} \oplus M_{\alpha_{23}} \oplus M_{\alpha_{123}}$$

In particular, the Hilbert series of  $\mathcal{B}(M^{x,a})$  is  $(2)_t^2(3)_t^2$ .



- There are 6 vertices with two orthogonally intersecting hyperplanes (the subdivision centers of the cuboctahedron squares) as in the examples  $a'$  the bounding hyperplanes  $x = H_{\alpha_{12}} \cap H_{\alpha_{23}}$ . It is associated

to the following restricted Nichols algebra  $\mathcal{B}(M^{x,a})$  in the category

$$\mathcal{C}_x = \frac{\mathcal{B}(M_{\alpha_{12}} \oplus M_{\alpha_{23}})}{\mathcal{B}(M_{\alpha_{12}} \oplus M_{\alpha_{23}})} \mathcal{YD}, \text{ where}$$

$$\mathcal{B}(M_{\alpha_{12}} \oplus M_{\alpha_{23}}) \cong \mathbb{C}[t_{\alpha_{12}}]/(t_{\alpha_{12}}^3) \otimes \mathbb{C}[t_{\alpha_{23}}]/(t_{\alpha_{23}}^3)$$

is a Nichols algebra of type  $A_1 \times A_1$ . The generating Yetter-Drinfel'd modules  $M^{x,a}$  is simple, the Nichols algebra of non-reduced type  $A_1$ :

$$M^{x,a} = M_{-\alpha_2} \oplus M_{\alpha_{12}} \oplus M_{\alpha_{23}} \oplus M_{\alpha_{123}} \quad [M^{x,a}, M^{x,a}] = M_{\alpha_{13}}$$

In particular, the Hilbert series of  $\mathcal{B}(M^{x,a})$  is  $(2)_t^4(3)_{t^2}$ .



- Again the  $(-1)$ -cell  $x = 0$  in the center is the trivial Nichols algebra  $1\mathbb{C}$  over any Weyl representative of the full Nichols algebra  $\mathcal{C}_x = \frac{\mathcal{B}(M^a)}{\mathcal{B}(M^a)} \mathcal{YD}$ .

## REFERENCES

- [AHS10] N. Andruskiewitsch, I. Heckenberger, H.-J. Schneider: *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010) 1493–1547
- [Ar62] S. Araki: *On root systems and an infinitesimal classification of irreducible symmetric spaces*, Journal of Mathematics, Osaka City University, Vol. 13, No. I. (Received May 8, 1962)
- [AY13] I. Angiono, H. Yamane: *The R-matrix of quantum doubles of Nichols algebras of diagonal type*, Preprint (2013), <http://arxiv.org/abs/1304.5752>.
- [BLS14] A. Barvels, S. Lentner, C. Schweigert: *Partially Dualized Hopf Algebras Have Equivalent Yetter-Drinfel'd Modules*, Preprint (2014), <http://arxiv.org/abs/1402.2214>
- [BC12] M. Barakat and M. Cuntz: *Coxeter and crystallographic arrangements are inductively free*, Adv. Math. **229** (2012), no. 1, 691–709.
- [CH09] M. Cuntz and I. Heckenberger: *Weyl groupoids with at most three objects*, J. Pure Appl. Algebra **213** (2009), no. 6, 1112–1128.
- [CH15] ———, *Finite Weyl groupoids*, J. Reine Angew. Math. **702** (2015), 77–108.
- [CMW15] M. Cuntz, B. Mühlherr, and C. Weigel: *Simplicial arrangements on convex cones*, Preprint (2015), 1–41.
- [CRT12] M. Cuntz, Y. Ren, and G. Trautmann: *Strongly symmetric smooth toric varieties*, Kyoto J. Math. **52** (2012), no. 3, 597–620.
- [Cun11] M. Cuntz, *Crystallographic arrangements: Weyl groupoids and simplicial arrangements*, Bull. London Math. Soc. **43** (2011), no. 4, 734–744.
- [Heck07] I. Heckenberger: *Lusztig isomorphisms for Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type*, Journal of Algebra **323** (2010) 2130–2182.

- [Heck09] I. Heckenberger: *Classification of arithmetic root systems*, Advances in Mathematics **220** (2009), no. 1, 59–124.
- [Heck10] I. Heckenberger: *Lusztig isomorphisms for Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type*, Journal of Algebra 323 (2010) 2130–2182.
- [HS10] I. Heckenberger, H.-J. Schneider: *Root systems and Weyl groupoids for Nichols algebras*, Proc. Lond. Math. Soc. 101 (2010) 623–654
- [HS13] I. Heckenberger, H.-J. Schneider: *Yetter-Drinfeld modules over bosonizations of dually paired Hopf algebras*, Adv. Math. 244 (2013) 354–394
- [HV14] I. Heckenberger and L. Vendramin: *A classification of Nichols algebras of semi-simple Yetter-Drinfeld modules over non-abelian groups*, Preprint 2014, arXiv:1412.0857.
- [HY08] I. Heckenberger and H. Yamane: *A generalization of Coxeter groups, root systems, and Matsumoto's theorem*, Math. Z. **259** (2008), no. 2, 255–276.
- [Kolb14] S. Kolb: *Quantum symmetric Kac-Moody pairs*, Adv. Math. 267 (2014), 395–469.
- [Len14a] S. Lentner: *New large-rank Nichols algebras over nonabelian groups with commutator subgroup  $\mathbb{Z}_2$* , Journal of Algebra **419** (2014), 1–33.
- [Len14b] S. Lentner: *Quantum affine algebras at small root of unity*, Preprint 2014, arXiv:1411.2959.
- [Let97] G. Letzter: *Subalgebras which appear in quantum Iwasawa decompositions*, Canadian Journal of Mathematics 49 (1997), 1206–1223.
- [Lusz90] G. Lusztig: *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math. Soc. 3 (1990), 257–296.
- [Lusz94] G. Lusztig: *Introduction to Quantum Groups*, Modern Birkhäuser Classics (2010), Springer.
- [MS00] A. Milinski and H.-J. Schneider: *Pointed indecomposable Hopf algebras over Coxeter groups*, New trends in Hopf algebra theory (La Falda, 1999) Contemporary Mathematics **267** (2000), 215–236.
- [NS95] M. Noumi and T. Sugitani: *Quantum symmetric spaces and related  $q$ -orthogonal polynomials*, Group theoretical methods in physics (Singapore) (A. Arima et. al., ed.), World Scientific (1995) pp. 28–40.
- [OT92] P. Orlik and H. Terao: *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer-Verlag, Berlin, 1992.
- [Rosso98] M. Rosso: *Quantum groups and quantum shuffles*, Invent. math. 133 (1998) p. 399–416.

MICHAEL CUNTZ, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK,  
 FAKULTÄT FÜR MATHEMATIK UND PHYSIK, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN  
 1, D-30167 HANNOVER, GERMANY

*E-mail address:* cuntz@math.uni-hannover.de

SIMON LENTNER, FACHBEREICH MATHEMATIK, BEREICH ALGEBRA UND ZAHLENTHEORIE,  
 UNIVERSITY HAMBURG

*E-mail address:* simon.lentner@uni-hamburg.de